SUBSCRIPTION
$2.00 PER YEAR
IN ADVANCE
SINGLE COPIES
25c.

All Business
Communications
should be addressed
to the
Editor and Manager

VOL. XV UNIVERSITY, LA, NOVEMBER, 1940. No. 2

Entered as second-class matter at University, Louisiana.

Published monthly excepting June, July, August, September, by LOUISIANA STATE UNIVERSITY, Vols. 1-8 Published as MATHEMATICS NEWS LETTER.

THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer. If approved for publication, the Committee will forward it to the Editor and Manager at Baton Rouge, who will notify the writer of its acceptance for publication. If the paper is not approved the Committee will so notify the Editor and Manager, who will inform the writer accordingly.

1. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

2. The name of the Chairman of each committee is the first in the list of the committee.

3. All manuscripts should be worded exactly as the author wishes them to appear in the MAGAZINE.

Papers intended for the Teacher's Department, Department of History of Mathematics, Bibliography and Reviews, or Problem Department should be sent to the respective Chairmen.

Committee on Algebra and Number Theory: L. E. Bush, W. Vann Parker, R. F. Rinehart.

Committee on Analysis and Geometry: W. E. Byrne, Wilson L. Miser, Dorothy McCoy, H. L. Smith, V. Thébault.

Committee on Teaching of Mathematics: Joseph Seidlin, James McGiffert.

Committee on Statistics: C. D. Smith, Irby C. Nichols.

Committee on Mathematical World News: L. J. Adams.

Committee on Bibliography and Reviews: H. A. Simmons.

Committee on Problem Department: R. C. Yates, E. P Starke.

Committee on Humanism and History of Mathematics: G. Waldo Dunnington.

PUBLISHED BY THE LOUISIANA STATE UNIVERSITY PRESS

Will You Be There?

Statements in the interest of the tri-organization mathematics meetings to be convened in Baton Rouge on the approaching December 30 were published in the April, May and October issues of the Magazine. These statements, addressed to nation-wide groups, were made by Vice-President Wren, Secretary Cairns, and President Evans, representing respectively, the National Council of Teachers of Mathematics, The Mathematical Association of America, and the American Mathematical Society.

In accord with unvarying practice, each of the three organizations definitely expects that a very large portion of those in attendance shall come from the territory relatively close to the meeting place. This expectation is wise in a country where distances are vast. The Caravan of new knowledge and fresh inspiration to tens of thousands of mathematical workers must, from year to year, be a *moving* Caravan!

It has been nine years since the mathematical Caravans came into the deep South—nine years since the Association and the Society held their Christmas meetings in New Orleans. Much water since has passed under the bridge. In a little while they come again—this time to Baton Rouge—not just two Caravans, but three, for the National Council of Teachers of Mathematics will be there.

Then, shall not we mathematics folk of the South, the Southwest, and the West do *our* part? If we shall be able to do so—even if it means sacrifice to come—shall we not come to these meetings?

Certainly not in the history of our Southland has appeared in its midst such wealth of opportunity to hosts of mathematical workers, in high school, college, university—as will be offered in the four-day programs prepared for the period from Monday, December 30, to Thursday, January 2, at University, Louisiana.

S. T. Sanders.

Logarithmic Properties of a Power of x

By W. E. SEWELL
Georgia School of Technology

1. *Introduction.* The purpose of this note is to describe the properties of the function

$$L_\alpha(x) = \int_1^x \frac{dx}{x^\alpha}, \quad 0<x<\infty, \quad \alpha \text{ real}, \tag{1.00}$$

and of the inverse function. Of course, for $\alpha=1$ we have the logarithmic and exponential functions whose properties are well known, and in our study we show how these properties generalize for non-unitary values of α. The study of the logarithmic and exponential functions based on the above definition is classical* and we use it as a pattern for our investigation.

It should be noted that the function $L_\alpha(x)$ is either transcendental or algebraic depending upon the value of α. However the properties in which we are interested do not require for their description a separate classification for these two types of functions. In fact our main purpose is simply to show how the classical properties look for arbitrary values of α.

Our point of view is primarily a consideration of a function of x with α real and fixed. The study could easily be extended to an investigation of the function of two independent variables or with α as the variable and x as the parameter. In this discussion we do not allow complex values for α.

2. *The function $L_\alpha(x)$.* From the definition several properties are immediate consequences.

$$L_\alpha(1)=0; \; L_\alpha(a)>0, \; a>1; \; L_\alpha(a)<0, \; 0<a<1. \tag{2.10}$$

The following theorem is easily proved:

Theorem 2.1. *The function $L_\alpha(x)$ is a single-valued and continuous function of x for each α. Furthermore it has a derivative which is*

$$L_\alpha'(x) = \frac{1}{x^\alpha}. \tag{2.11}$$

*See, e. g., Osgood, *Lehrbuch der Funktionentheorie*, 5th ed., Vol. 1, Leipzig, 1928. pp. 584-588.

As a matter of fact this theorem is an immediate consequence of the well known properties of the definite integral.

Corollary 2.10. *$L_\alpha(x)$ is monotonic, and increases with x.*

Theorem 2.2. *The function $L_\alpha(x)$ is a continuous function of α and has a derivative with respect to α which is*

$$\frac{\partial L_\alpha(x)}{\partial \alpha} = -\int_1^x \frac{\operatorname{Log} x\, dx}{x^\alpha}.$$

Theorem 2.3. *If $L_\alpha(x) = L_\alpha(y)$, then $x = y$.*

From the mean value theorem we have

$$L_\alpha(y) = L_\alpha(x) + (y-x)L_\alpha'[x+\theta(y-x)]$$

and since the derivative of $L_\alpha(x)$ is never zero, $y-x=0$.

Theorem 2.4. *We have*

$$L_\alpha(xy) = \frac{(1+y^{1-\alpha})L_\alpha(x) + (1+x^{1-\alpha})L_\alpha(y)}{2}. \tag{2.12}$$

For

$$\int_1^{xy} \frac{dt}{t^\alpha} = \int_1^x \frac{dt}{t^\alpha} + \int_x^{xy} \frac{dt}{t^\alpha}. \tag{2.13}$$

Let $t = x\tau$ in the second integral on the right, then

$$\int_1^{xy} \frac{dt}{t^\alpha} = \int_1^x \frac{dt}{t^\alpha} + x^{1-\alpha} \int_1^y \frac{d\tau}{\tau^\alpha},$$

Consequently

$$\int_1^{xy} \frac{dt}{t^\alpha} = \int_1^x \frac{dt}{t^\alpha} + x^{1-\alpha} \int_1^y \frac{dt}{t^\alpha}, \text{ or} \tag{2.14}$$

$$L_\alpha(xy) = L_\alpha(x) + x^{1-\alpha}L_\alpha(y). \tag{2.15}$$

Since x and y are interchangeable we have also

$$L_\alpha(xy) = y^{1-\alpha}L_\alpha(x) + L_\alpha(y). \tag{2.16}$$

By adding (2.15) and (2.16) we obtain the result of the theorem. Another form is

$$L_\alpha(x) + L_\alpha(y) = L_\alpha(xy) + \tfrac{1}{2}\{(1-x^{1-\alpha})L_\alpha(y) + (1-y^{1-\alpha})L_\alpha(x)\}.$$

Putting $y=1/x$ in (2.12) and using the fact that $L_\alpha(1)=0$ we obtain the corollary.

Corollary 2.40. *We have*

$$L_\alpha\left(\frac{1}{x}\right)=-\frac{1}{x^{1-\alpha}}L_\alpha(x). \tag{2.17}$$

We prove the following result by mathematical induction:

Theorem 2.5. *We have*

$$L_\alpha(x^n)=\left(\sum_{k=0}^{n-1}(x^k)^{1-\alpha}\right)L_\alpha(x) \tag{2.18}$$

$$=\frac{1-(x^{1-\alpha})^n}{1-x^{1-\alpha}}L_\alpha(x),\ x\neq 1,\ \alpha\neq 1,$$

where n is an integer.

The theorem is true for $n=1$ obviously and for $n=2$ we have from (2.12)

$$L_\alpha(x^2)=(1+x^{1-\alpha})L_\alpha(x),$$

which is precisely (2.18). Now suppose the result is true for $n=m-1$, then we have

$$L_\alpha(x^{m-1})=\frac{1-(x^{1-\alpha})^{m-1}}{1-x^{1-\alpha}}L_\alpha(x)$$

and by (2.12)

$$L_\alpha(x^n)=\frac{(1+x^{1-\alpha})L_\alpha(x^{m-1})+\{1+(x^{m-1})^{1-\alpha}\}L_\alpha(x)}{2}$$

$$=\frac{(1+\alpha^{1-\alpha})\left(\dfrac{1-(x^{1-\alpha})^{m-1}}{1-x^{1-\alpha}}\right)L_\alpha(x)+\{1+(\alpha^{1-\alpha})^{m-1}\}L_\alpha(x)}{2}$$

$$=\left\{\frac{\begin{array}{c}1+x^{1-\alpha}-(x^{1-\alpha})^{m-1}-(x^{1-\alpha})^m\\+1+(x^{1-\alpha})^{m-1}-x^{1-\alpha}(x^{1-\alpha})^m\end{array}}{2(1-x^{1-\alpha})}\right\}L_\alpha(x).$$

Thus $$L_\alpha(x^n)=\frac{1-(x^{1-\alpha})^n}{1-x^{1-\alpha}}L_\alpha(x),\quad x\neq 1,\quad \alpha\neq q,$$

for n a positive integer. For n a negative integer we have, letting $m=-n$,

$$L_\alpha(x^n)=L_\alpha\left(\frac{1}{x^m}\right)=-\frac{1}{(x^m)^{1-\alpha}}L_\alpha(x^m)$$

$$=-\frac{1}{(x^{1-\alpha})^m}\cdot\frac{1-(x^{1-\alpha})^m}{1-x^{1-\alpha}}L_\alpha(x)$$

$$=-(x^{1-\alpha})^n\left\{\frac{(x^{1-\alpha})^n-1}{(x^{1-\alpha})^n}\right\}\frac{L_\alpha(x)}{1-x^{1-\alpha}},$$

and the result is established for every integer.

Theorem 2.6. *We have*

$$L_\alpha(+\infty)=+\infty,\quad 0<\alpha\leqq 1, \tag{2.19}$$

$$L_\alpha(+\infty)=\frac{+1}{\alpha-1},\quad \alpha>1, \tag{2.20}$$

$$L_\alpha(0^+)=-\infty,\quad \alpha\geq 1, \tag{2.21}$$

$$L_\alpha(0^+)=\frac{1}{\alpha-1},\quad 0<\alpha<1, \tag{2.22}$$

$$\lim_{x\to\infty}\frac{L_\alpha(x)}{x^{1-\alpha}}=\frac{1/x^\alpha}{(1-\alpha)x^{-\alpha}}=\frac{1}{1-\alpha},\quad 0<\alpha<1. \tag{2.23}$$

Since $L_\alpha(x)$ is continuous we may consider a sequence of points x_n and establish the limit of the function by its behavior along this sequence. For $\alpha<1$ by (2.18) we have

$$L_\alpha(2^n)=\left[\sum_{k=0}^{n-1}(2^k)^{1-\alpha}\right]L_\alpha(2)>nL_\alpha(2),$$

and since $L_\alpha(2)>0$ we have (2.19). By (2.23) and (2.17) we obtain (2.22). We have (2.21) directly from (2.19) and (2.18), and using (2.21) in conjunction with (2.17) we obtain (2.20).

Theorem 2.7. *For $\alpha<1$, $L_\alpha(x)=C_1$, $1/(\alpha-1)\leqq C<\infty$, has one and only one solution. For $\alpha>1$, $L_\alpha(x)=C$, $C<1/(\alpha-1)$, has one and only one solution. For $\alpha=1$, $L_\alpha(x)=C$, where C is arbitrary, has one and only one solution.*

The first result follows from (2.22) and (2.19), the second from (2.21) and (2.20), and the third from (2.21) and (2.19).

Theorem 2.8. *The function $L_\alpha(x)$ is the solution of the differential equation*

$$\frac{dy(x)}{dx}=\frac{1}{x^\alpha},$$

with the boundary condition $y(1)=0$.

This result follows directly from the definition and (2.11).

3. The inverse function. We begin with

Theorem 3.0. *The inverse function $y=E_\alpha(x)$, where $x=L_\alpha(y)$, is single-valued and continuous for the admissible values of the argument (see Theorem 2.6), and is positive throughout its range. Furthermore it possesses a derivative which is given by the formula.*

$$E'_\alpha(x)=[E_\alpha(x)]^\alpha \tag{3.10}$$

This result follows directly from the mean value theorem for derivatives. For (3.10) we have $E_\alpha(L_\alpha(y))=y$

$$E'_\alpha L'_\alpha=1,$$

$$E'_\alpha(x)=\frac{1}{L'_\alpha(y)}=\frac{1}{1/y^\alpha}=y^\alpha=[E_\alpha(x)]^\alpha$$

Theorem 3.1. *The function $E_\alpha(x)$ is a monotone increasing function of x. Hence if $E_\alpha(x)=E_\alpha(y)$, then $x=y$. Moreover $E_\alpha(x)=C$* (see theorem 2.8 for restrictions on C) *has one and only one solution.*

This follows directly from the results of Theorem 3.0.

Theorem 3.2. *We have.*

$$E_\alpha(+\infty)=\infty,\quad 0<\alpha\leqq 1, \tag{3.11}$$

$$E_\alpha\left(\frac{1}{\alpha-1}\right)=+\infty,\quad \alpha>1, \tag{3.12}$$

$$E_\alpha(-\infty)=0,\quad \alpha\geqq 1, \tag{3.13}$$

$$E_\alpha\left(\frac{1}{\alpha-1}\right)=0,\ 0<\alpha<1, \tag{3.14}$$

$$E_\alpha(0)=1. \tag{3.15}$$

These are the direct consequences of theorem 2.7 and the fact that $L_\alpha(1)=0$.

Theorem 3.3. *The functions $E_\alpha(x)$ satisfies the functional equation*

$$E_\alpha[x+y+\epsilon(x,y)] = E_\alpha(x)E_\alpha(y), \tag{3.16}$$

where

$$\epsilon(x,y) = -\tfrac{1}{2}\{(1-E_\alpha(x)^{1-\alpha})y+(1-E_\alpha(y)^{1-\alpha})x\} \tag{3.17}$$

Let $\left.\begin{matrix} E_\alpha(x)=\xi \\ x=L_\alpha(\xi) \end{matrix}\right\}, \quad \left.\begin{matrix} E_\alpha(y)=\eta \\ y=L_\alpha(\eta) \end{matrix}\right\}$

Then $$x+y=L_\alpha(\xi)+L_\alpha(\eta)$$

From (2.12) we have

$$L_\alpha(\xi)+L_\alpha(\eta)=L_\alpha(\xi\eta)+(1-\xi^{1-\alpha})L_\alpha(\eta)$$

$$L_\alpha(\xi)+L_\alpha(\eta)=L_\alpha(\xi)\eta+(1-\eta^{1-\alpha})L_\alpha(\xi).$$

Therefore

$$L_\alpha(\xi)+L_\alpha(\eta)=L_\alpha(\xi\eta)+\tfrac{1}{2}\{(1-\xi^{1-\alpha})L_\alpha(\eta)+(1-\eta)^{1-\alpha})L_\alpha(\xi)\}$$

consequently

$$E_\alpha(x+y)=E_\alpha\left\{\begin{matrix} \tfrac{1}{2}[(1+E_\alpha(y)^{1-\alpha})x+(1-E_\alpha(x)^{1-\alpha}y] \\ +\tfrac{1}{2}[(1-E_\alpha(x)^{1-\alpha})y+(1-E_\alpha(y)^{1-\alpha})x] \end{matrix}\right\}$$

A better form is

$$E_\alpha[x+y-\tfrac{1}{2}\{(1-E_\alpha(x)^{1-\alpha})y+(1-E_\alpha(y)^{1-\alpha})x\}]=E_\alpha(x)E_\alpha(y)$$

or let $\epsilon(x,y)=-\tfrac{1}{2}\{(1-E_\alpha(x)^{1-\alpha})y+(1-E_\alpha(y)^{1-\alpha})x\}$,

then $E_\alpha[x+y+\epsilon(x,y)]=E_\alpha(x)E_\alpha(y).$

The Angle Ruler, the Marked Ruler and the Carpenter's Square*

By Robert C. Yates
Louisiana State University

The three instruments discussed herein do not belong to the so-called classical set nor are they used in the classical sense. Although no justification need be made for this departure, I do wish to point out that the manner in which these tools are used is no more difficult physically than that governing the classical straightedge. Since the constructional possibilities of a tool depend entirely upon its exact nature and the precise manner in which it is used, these things are made the focal point of attention.

There is a connecting thread among these three apparently unrelated tools which brings them together in this discussion. This connection is brought to light at the end of the paper.

THE ANGLE RULER

Definition: The Angle Ruler is an unmarked plane instrument having two straight edges of indefinite length meeting at an arbitrary constant angle.†

Use: The Ruler shall be used in two ways:

I. so that a single edge is in contact with two fixed points of the plane, and

II. so that a different edge is in contact with each of two such points or with a point and a line.

Used in this fashion, the Angle Ruler is equivalent to the euclidean straightedge and compasses. As given by Steiner, this equivalence is established if it is possible to make the following seven fundamental constructions:

(1) To find the intersection of two lines,

(2) To draw through a point a line parallel to a given line,

(3) To draw through a point a line perpendicular to a given line,

*Read in part at Lafayette College, March 21, and before the Southeastern Section of the Association, Athens, Georgia, March 29, 1940.

†This includes both the Parallel and Right-angle Rulers as special cases.

(4) To extend a line segment its own length,

(5) To construct the fourth proportional to three given line segments,

(6) To find the intersections of a given line and hypocircle,*

(7) To find the intersections of two hypocircles.

Excepting the first which is obvious, these constructions follow.

(2) Given the line L and the external point P. With one edge of the Ruler coincident with L draw the line K. Now place one edge

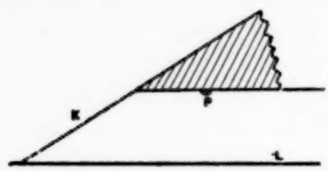


FIG. 1

upon K so that the other edge passes through P. This determines the parallel to L through P.

(3) The perpendicular from P to L is obtained by forming the rhombus with L as one diagonal, placing the Ruler so that one edge

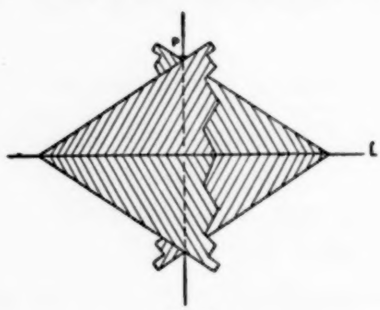


FIG. 2

coincides with L while the other passes through P as shown in Figure 2. The four positions of the Ruler determine the perpendicular.

(4) To extend the segment AB, Figure 3, place an edge of the Ruler in contact with each point A, B and mark a position V of the vertex. Draw through V a parallel to AB, and through A a parallel AV' to BV (by turning the Ruler upside down). With the Ruler in

*The word *hypocircle* (contraction of "hypothetical circle") is used to signify the circle defined only by its center O and a point A upon its circumference; thus: $O(A)$.

the new position $AV'C$ the remaining edge cuts AB produced in C such that $CA = AB$.

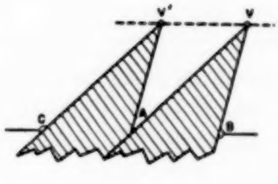


FIG. 3

(5) The construction of a fourth proportional x to the segments a, b, c is simply a matter of producing similar triangles

(6) The intersections of a given line L and a hypocircle $O(A)$ are found as follows. Extend the segment AO to A' so that AA' is the

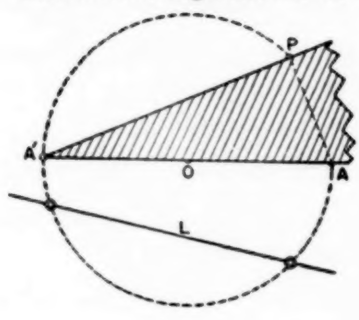


FIG. 4

diameter of the circle. Place the Ruler with vertex at A' and one edge along AA'. The line determined by the other edge meets its perpendicular from A in P, a point of the circle. Now move the Ruler with these edges touching A and P until the vertex falls upon L. This gives the required intersections.

(7) The intersections of two hypocircles $O(A)$ and $O'(K)$ are the same as the intersections of one of them and their radical axis.

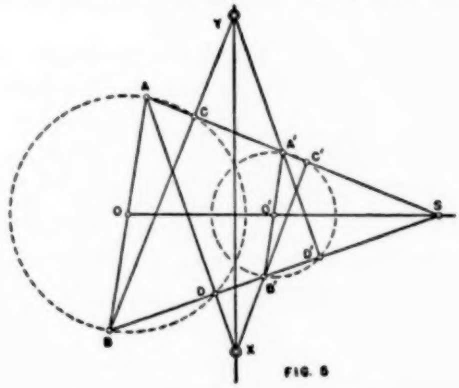


FIG. 5

In the manner of (6), locate another point A' on circle $O'(K)$ such that $O'A'$ is parallel to OA. The center of similitude S of the given circles is then the intersection of OO' and AA'. Draw SAA' and find its other intersections, C and C', with the given circles. Locate B and B', the extremities of the diameters AOB and $A'O'B'$. Draw SBB' and find its other intersections, D and D', with the given circles. Now AD and $B'C'$ meet in X; BC and $A'D'$ meet in Y; and XY is the radical axis. This construction then reduces to (6).

Thus the Angle Ruler is a quadratic tool; that is, it is capable of making all geometrical constructions whose analytic formulation leads to equations of the second degree whose coefficients represent possessed lengths and whose roots are real.

THE MARKED RULER

Definition: The Marked Ruler is a single straightedge of indefinite length upon the edge of which two points P, Q are marked. We shall take the distance PQ as the unit of measure.

Use: The Ruler shall be used in three ways:

I. To establish the line upon two given points and to mark upon this line successive unit lengths;

II. To fix Q at a given point of the plane and rotate the Ruler until P falls upon a given line;

III. With the straightedge passing through a given point of the plane, to move Q along a given line until P falls upon a second given line.

We consider the fundamental Steiner constructions under these privileges, first under I:

(2) The line through F parallel to L is obtained by applying two consecutive units, PQ, to L to establish a bisected segment ABC.

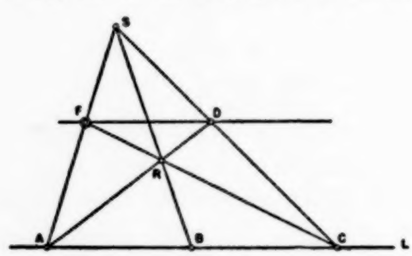


FIG. 6

Draw AF and CF and upon AF select an arbitrary point S. Draw SB and SC, the former meeting FC in R. Now AR meets SC in D and FD is parallel to AC. This will be recognized as the construction of the fourth harmonic paired with B in the set $(AC;B\infty)$.

(3) The perpendicular from F to L is obtained by locating the points A, B, C on L such that $AB=BC=1$; then the two additional

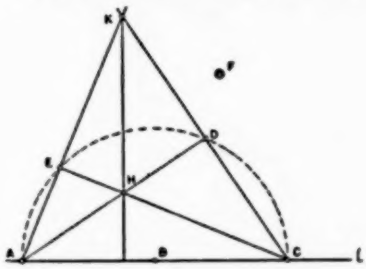


FIG. 7

points D, E, not on L, such that $BD=BE=1$. These four points, A, C, D, E, lie on the unit circle with B as center. Accordingly, AD and CE meet in the orthocenter H of triangle AKC, where K is the intersection of AE and CD. Thus KH is perpendicular to L. The line through F parallel to KH is the desired perpendicular.

(4) The construction of the fourth proportional follows immediately from (2).

(5) The extension of the segment AB is obtained by locating X, Y on AB such that $XB=BY=1$; erecting the perpendicular to

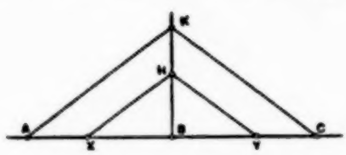


FIG. 8

AB at B which meets an arbitrary line through X in H. The line through A parallel to XH meets BH in K. The parallel to HY through K determines C such that $AB=BC$.

The sixth and seventh Steiner constructions are not possible under assumption I. However, II enlarges the powers of the Marked Ruler to include them; thus

(6) To find the intersections of L and the hypocircle $O(A)$, where $OA\neq 1$, construct the perpendicular OC to L and let OA' upon

OA equal the unit distance. Draw AC and its parallel $A'C'$. Now draw L' through C' parallel to L and determine its intersections X', Y'

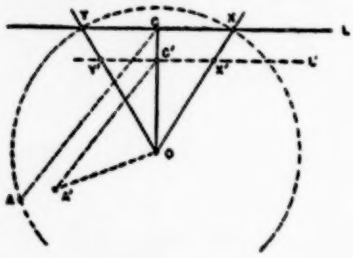


FIG. 9

with the unit circle, center at O. The lines OX', OY' determine the required intersections X, Y.

(7) The intersections of two given hypocircles reduces to (6) on constructing their radical axis. This is similar to the discussion under the Angle Ruler and is omitted here.

Accordingly, *the Marked Ruler used under I and II is a quadratic tool and is equivalent to the straightedge and compasses. It is thus capable of the rational operations of addition, subtraction, multiplication, and division of possessed segments together with the irrational operation of extracting the square root of such segments.*

Consider the quartic

$$x^4+ax^3+bx^2+cx+d=0, \tag{1}$$

where a, b, c, d represent possessed line segments. We shall show that this quartic is reducible to a resolvent cubic by means of the Marked Ruler used under I and II. If we let $x=y-a/4$, equation (1) reduces to

$$y^4+Ay^2+By+C=0, \tag{2}$$

where A, B, C are rational functions of a, b, c, d and thus constructible under I and II. Now let

$$y=\pm\sqrt{z_1}\pm\sqrt{z_2}\pm\sqrt{z_3}$$

and impose upon z_i the conditions:

$$\sqrt{z_1}\sqrt{z_2}\sqrt{z_3}=-B/8$$
$$z_1+z_2+z_3=-A/2$$
$$z_2z_3+z_3z_1+z_1z_2=(A^2-4C)/16.$$

Then z_i will be roots of the cubic:

$$z^3+Az^2/2+(A^2-4C)z/16-B^2/64=0, \quad \text{or}$$

$$z^3+Dz^2+Ez+F=0. \tag{3}$$

The coefficients in (3) are also rational functions of a, b, c, d and thus constructible in the same manner. This is a resolvent cubic and its roots lead to those of (1). It is desirable, however, to make some further reductions.

Let $z=s-D/3$. Equation (3) becomes:

$$s^3+Hs+K=0,$$

where $H=E-D^2/3$, $K=2D^3/27-DE/3+F$. Now let $s=Kt/H$. The last equation reduces to

$$\boxed{t^3+m(t+1)=0,} \tag{4}$$

where $m=H^3/K^2$.

Retracing the steps of these reductions, it is evident that *the general quartic is reducible to a solvent cubic dependent upon a single constant, wherein the only operations upon the given coefficients are those that are equivalent to constructions of the Marked Ruler under I and II.*

The cubic (4) may be solved by setting $t=u+v$ and restricting u and v to

$$u^3+v^3=-m, \quad uv=-m/3.$$

This produces the resolvent quadratic:

$$27u^6+27mu^3-m^3=0,$$

one of whose solutions is

$$u^3=(m/2)[-1+\sqrt{(1+4m/27)}]=R. \tag{5}$$

Since the discriminant of (4) is

$$\Delta=-m^2(27+4m),$$

the roots t are

(a) one real, two complex if $m>-27/4$;

(b) all real if $m\leq-27/4$.

If (a) holds, the real root is found from the real cube root of R; that is,

$$t_1=\sqrt[3]{R}-m/3\sqrt[3]{R}.$$

If (b) holds, the roots may be obtained by trisecting a certain angle as follows: Consider the identity:

$$4\cos^3\theta - 3\cos\theta + \cos 3\theta = 0$$

and let $2\cos\theta = r$. We have the Trisection Equation

$$r^3 - 3r + 2a = 0, \tag{6}$$

where $a = \cos 3\theta$, $|a| \leq 1$. Now in equation (4), if we let

$$t = r\sqrt{(-m/3)},$$

a real transformation since $m \leq -27/4$, we have:

$$r^3 - 3r - 3\sqrt{(-3/m)} = 0. \tag{7}$$

This will be identical with (6) if

$$-2 \leq -3\sqrt{(-3/m)} \leq +2.$$

The values of m that satisfy this inequality are $m \leq -27/4$. But this is just the condition that equation (4) have all real roots.

Accordingly, *the real solutions of the general cubic can be obtained either from the trisection of a certain angle or from the cube root of a certain number.*

Note that if the values $\sqrt[3]{R}$ under (a), or r under (b), can be found as line segments, the process leading back to the original roots t of (4) or z of (3) demands nothing more than constructions that are possible by the Marked Ruler under I and II.

Consider now the Marked Ruler employed under privileges I, II, and III.

Trisection: Let $AOB = 3\theta$, Figure 10, be a given angle whose cosine is a. Construct PK and KH parallel and perpendicular re-

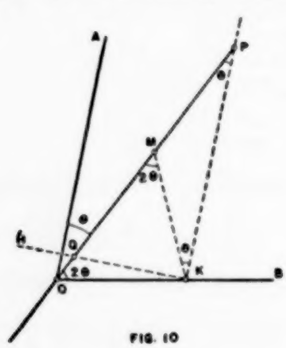


FIG. 10

spectively to OA, where $OK=1/2$ (the bisected unit of the ruler). Move the Ruler so that Q travels along KH, while its edge passes through O, until P meets the line KP. In this position the angle AOB is trisected. For, if M be the midpoint of PQ, then

$$\measuredangle AOP = \measuredangle OPK = \measuredangle PKM = \theta,$$

and thus
$$\measuredangle KMO = \measuredangle KOM = 2\theta.$$

There are two positions of the Ruler other than the one shown which determine the remaining roots of equation (6). These give the trisections of the two angles, $360°+3\theta$ and $720°+3\theta$, "induced" from 3θ. Note that the path of P is the well known *conchoid* having HK as its asymptote.

Cube Root: For the extraction of the cube root of R, proceed as follows. Upon a selected line let $XZ=ZK=R/4$ and locate O so that $OX=OZ=1$. Draw KO and its parallel MX. Move the Ruler so

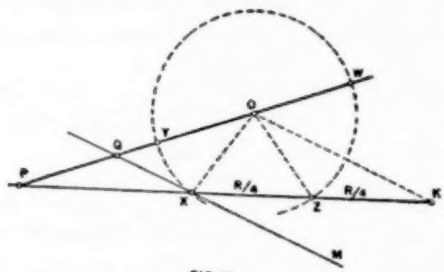


FIG. 11

that Q travels along MX with its edge passing through O until P falls upon XZ produced. In this position,

$$PX = \sqrt[3]{R}.$$

For, if $PX=x$, $PY=y$, from similar triangles,

$$(PX)/(PQ) = (XK)/(QO) \quad \text{or} \quad x = R/2y.$$

From the secant property of the circle:

$$(PX)(PZ) = (PY)(PW), \quad \text{or} \quad x(x+R/4) = y(y+2).$$

Eliminating y between these two equations, we have:

$$(4x+R)(x^3-R)=0$$

which has the real solution*

$$x = \sqrt[3]{R}.$$

*The root $x=-R/4$, which gives the position of the Ruler through O and Z plays no role here.

Thus *the Marked Ruler under I, II, III is capable of constructing all real roots of the general quartic whose coefficients represent possessed lengths, eventually either as the trisection of a certain angle or as the cube root of a certain segment.*

THE CARPENTER'S SQUARE

Definition: The Carpenter's Square is an unmarked ruler with parallel edges, one of whose legs is terminated and at right angles to the other. The terminated edge BD has marked upon it the bisecting point P. We take the width of the ruler as the unit of measure.

Use: I. Since either leg is a special Angle Ruler, it is equivalent to the straightedge and compasses if used in the manner indicated for this tool.

II. It will also be used so that a corner travels along a fixed line while an edge passes through a fixed point until another corner, or P, falls upon a fixed line.

The tool is obviously *quadratic* under I. But the additional privilege II makes it *quartic*. To prove this we need only demonstrate that it is both a trisector and a cube root extractor.

Trisection: Let $BOF = 3\theta$ be a given angle. Draw DD' parallel to OF at a unit distance (the width of the Ruler). Allow D to move along DD' with the inner edge of the Ruler passing through O until B

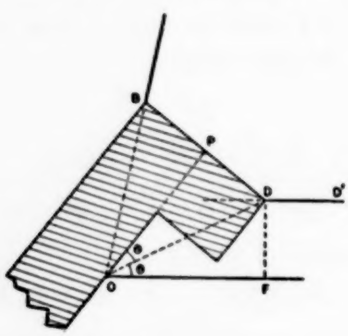


FIG. 12

falls upon OB as shown. The angle is then trisected by the lines OD and OP. For, since $BP = PD = DF = 1$, the right triangles OPB, OPD, and OFD are all congruent with equal angles θ at O. As in the Marked Ruler, two other positions are possible. Note that the bisecting point P does not play an essential role here.

Cube Root: Let the outer edge of the Square pass through A, Figure 13, while the corner D travels along the line CD, two units distant from A. We seek the path of P. To obtain its equation, take

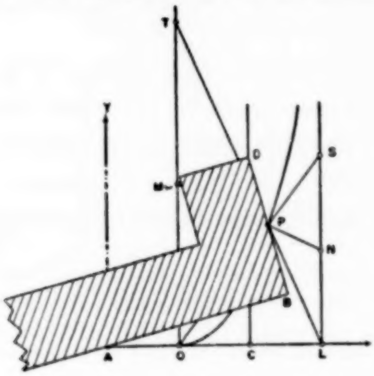


FIG 13

the perpendicular from A to CD as X-axis and the perpendicular to this at A for Y-axis. Then $BD = AC = 2$; $AB = DC$. Let the coordinates of P be (x,y); those of B: $(2x-2, 2z)$. Then, since $BP = PD$, the coordinates of D are $(2, 2y-2z)$. Now since $AB = DC$:

$$(x-1)^2 + z^2 = (y-z)^2 \quad \text{or} \quad (x-1)^2 = y(y-2z),$$

and since AB is perpendicular to BD:

$$(x-1)(x-2) = z(y-2z).$$

Combining these two equations to eliminate z, we have:

$$y^2 = (x-1)^3/(3-x),$$

the equation of the *cissoid* having $x=3$ as asymptote and cusp at $(1,0)$.

For the extraction of the cube root of a segment R, draw through the cusp O a line parallel to the Y-axis and upon it mark $OM = R$ and $OT = 2R$. Draw LT and move the square through A as indicated above until P falls upon LT. Draw the line MPN. Then

$$LN = \sqrt[3]{R}.$$

For, the equation of the curve may be written in the form:

$$[y/(x-1)]^3 = y/(3-x).$$

Now a line OS, $y/(x-1) = m$, through the cusp meets the curve in a point P whose coordinates satisfy

$$m^3 = y/(3-x).$$

But this equation may be thought of as representing the line through (3,0) and P; that is, the line LT. But LT meets OT in the point $(1,2m^3)$ and, since $LS=2m$, $LN=(LS)/2$, $OM=(OT)/2$, then

$$(LN)^3=OM=R.$$

Thus *the Carpenter's Square under I and II is capable of constructing all real roots of the general quartic whose coefficients represent possessed lengths, eventually either as the trisection of a certain angle or as the cube root of a certain segment.*

In conclusion, I wish to point out the connection among these three tools.

The Angle Ruler may be defined as:

Two lines fixed with reference to each other on a point.

The dual of this defines the Marked Ruler:

Two points fixed with reference to each other on a line.

The use to which the Angle Ruler is put to make it a quadratic tool:

> *To move so that its two lines (edges) are in contact with two fixed points of the plane until its point (vertex) coincides with a fixed line of the plane,*

dualizes into the use to which the Marked Ruler is put to make it a quartic tool:

> *To move so that its two points are in contact with two fixed lines of the plane until its line coincides with a fixed point of the plane.*

The Carpenter's Square may be defined as an extended line to which there is rigidly attached a line segment and it is, in this sense, a generalization of the Marked Ruler.

BIBLIOGRAPHY

1. ADLER, A., *Theorie der geometrischen Konstruktionen*, Leipzig, (1906).
2. ARCHIBALD, R. C., American Mathematical Monthly, (1918), pp. 358-360. (This is the source of many references given here.)
3. AUBRY, Journal de Mathématiques spéciales, (1895 and 1896).
4. BUSSEY, W. H., *Geometric Constructions Without the Classical Restriction to Ruler and Compasses*, American Mathematical Monthly, (1936), pp. 265-280.
5. CONCINA, U., *Resoluzione dei problemi fondamentali relativi al trasporto delle figure piane colla riga a due orli paralleli*, Il bolletino di mathematiche e di scienze fisiche e naturali, Bologna, (1901), pp. 225-237.

6. DESCARTES, R., *La Géometrie*, livre II. Oeuvres de Descartes, publiées par C. Adam et P. Tannery, V. 6, Paris (1902), p. 391.

7. DE TILLY, *Sur la géométrie de la règle*, Nouvelle correspondence mathématique, tome 3, (1877), pp. 204-208; tome 5, (1879), pp. 439-442; tome 6, (1880), pp. 34-35.

8. ENRIQUES, F., *Fragen der Elementargeometrie*, Leipzig, (1923), pp. 212-215.

9. FOURREY, E., *Procédés Originaux de Constructions Géométriques*, Librarie Vuibert, Paris, (1924).

10. HILBERT, D., *The Foundations of Geometry*, Chicago, (1910).

11. HUDSON, H. P., *Ruler and Compasses*, London, (1916).

12. LONGCHAMPS, G. DE, *Essai sur la géométrie de la règle et de l'équerre*, Paris, (1890), pp. 232-235.

13. MARENCHI, C., *Geometria della riga a due orli paralleli*, Il bolletino di mathematiche e di scienze fisiche e naturali, Bologna, (1900-1901), Vol. 2, pp. 129-145.

14. MASCHERONI, L., *Problemi per gli agrimensori con varie soluzioni*, In Pavia, MDCCXCIII.

15. PONCELET, J. V., *Traite des propriétés projectives des figures*, Paris, (1822).

16. ROBERTS, S., *On the Mechanical Description of Some Species of Circular Curves of the Third and Fourth Degrees*, Proceedings of the London Mathematical Society, II, (1869), pp. 125-136.

17. SERVOIS, F. J., *Solutions peu connues de différens problèmes de géométrie-pratique. . . .*, A. Metz, An XII, (1804).

Members of the Editorial Board of this journal acknowledge with appreciation the assistance of the following mathematicians, who have acted as referees for papers submitted for publication in NATIONAL MATHEMATICS MAGAZINE, *throughout the year 1939-40, Volume XIV:*

Professor R. H. Cameron, Mass. Inst. of Tech.; Professor J. Wm. Peters, University of Illinois; Dr. K. L. Palmquist, Southern Methodist University; Professor F. L. Brooks, Kent State University (Ohio); Professor E. H. Hanson, North Texas State Teachers College; Dr. Grace Shover, Carleton College; Professor H. P. Thielman, College of St. Thomas; Professor C. J. Blackall, College of St. Thomas; Dr. J. M. Dobbie, Case School of Applied Science; Professor E. T. Browne, University of North Carolina; Professor B. W. Jones, Cornell University; Professor F. E. Johnston, George Washington University, (Washington, D. C.). Professor A. E. Landry, Catholic University of America; Professor F. C. Gentry, Louisiana Polytechnic Institute; Professor Watson M. Davis, Illinois Institute of Technology.

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington

Hypatia of Alexandria

By A. W. Richeson
University of Maryland

The first woman mathematician regarding whom we have positive knowledge is the celebrated mathematician-philosopher Hypatia. The exact date of her birth is not known, but recent studies indicate that she was born about A. D. 370 in Alexandria. This would make her about 45 years of age at her death. Hypatia, it seems, was known by two different names, or at least by two different spellings of the same name; the one, Hypatia; the other, Hyptachia. According to Meyer,[1] there were two women with the same name living at about this time; Hypatia, the daughter of Theon of Alexandria; the other, the daughter of Erythrios. Hypatia's father was the well-known mathematician and astronomer Theon, a contemporary of Pappus, who lived at Alexandria during the reign of Emperor Theodosius I. Theon, the director of the Museum or University at Alexandria, is usually considered as a philosopher by his biographers.

Hypatia's biographers have given us but little of her early personal history. We know that she was reared in close touch with the Museum in Alexandria, and we are probably safe in assuming that she received the greater part of her early education from her father. If we are to judge from the records which the historians have left us, we would conclude that her early life was uneventful. It would seem that she spent the greater part of her time in study and reading with her father in the Museum.

Suidas[2] and Socrates,[3] as well as others who lived at the same time, lead us to believe that Hypatia possessed a body of rare beauty and grace. They attest not only to her beauty of form and coloring, but each and every one speaks just as highly of the beauty of her charac-

[1] Meyer, Wolfgang Alexander, *Hypatia von Alexandria*, Heidelberg, 1886, pp. 52.
[2] Socrates, *The Ecclesiastical History*, Trans. by Henry Bohn, London, 1853.
[3] Suidae, *Lexicon, Lexicographi Graeci*, Vol. I, Pars IV, ed. Ada Adler, Lipsiae, 1935.

Gasparo's portrait of
HYPATIA
(reproduced by permission of the copyright owners of "Little Journeys")

Medallion of Hypatia in the Introduction to Halma's edition of Theon's "Commentary". (Artist unknown)

ter. In the absence of a life painting of Hypatia we must depend upon the conception of others for a picture of the philosopher. In the introduction to his edition of Theon's Commentary[1] Halma has given us a short biography of Hypatia. On the title-page there is a medallion which gives his conception of the philosopher. Meyer feels that this drawing is unfortunate, as he does not believe it gives a true impression of the woman Hypatia. Charles Kingsley, on the other hand, in his novel Hypatia has written a vivid description of his impression of the philosopher.

If we are to believe the historians as to her beauty, we would expect that she was eagerly sought after in marriage. This apparently was the case: her suitors included not only outsiders, but many of her students as well. The question of her marriage, however, leads us to one of the controversial points of her life. Suidas states she was the wife of the philosopher Isidorus; then 25 lines later, he states she died a virgin. This apparent contradiction has been explained in several ways by later writers.

Toland[2] believes she was engaged to Isidorus before she was murdered, but was never married. Hoche[3] is of the opinion that the mistake arose from Suidas' abstract of the works of Damascius, a conclusion which Meyer does not believe to be true, pointing out that he found on the margin of one of Photius' works the statement, "Hypatia, Isidore uxor." Since Photius transcribed Hesychius' works, it is possible that the error arose in this manner. The evidence against such a marriage is further substantiated by the fact that Damascius states that Isidorus was married to a woman named Danna and had a child by this wife. Another fact which should be taken into consideration is that Proclus was much older than Isidorus: it has been pretty definitely established that Proclus was born about 412, and, since Hypatia's death occurred in the year 415, it would be impossible for Hypatia to have been the wife of Isidorus. The present writer is inclined to agree with Meyer that the mistake arose in Photius' transcription of Hesychius' work and that Hypatia was not married at any time in her life.

The second controversial point is the question of her death. In studying the statements made by many of the historians in regard to her death it seems desirable to review the murder in relation to the events which had happened previously. It is necessary for us to

[1] *Theon d'Alexandrie, Commentaire sur le livre III de l'Almageste de Ptolemee*, ed. Halma, Paris, 1882.

[2] Toland, John. *Tetradymus*, London, 1720, pp. 101-136.

[3] Hoche, Richard. *Hypatia die Tochter Theons*, Philologus, Fünfzehnter Jahrgang, Göttingen, 1869, pp. 435-474.

investigate not only Hypatia's relation to paganism, but also the relation between Cyril, the Christian bishop at Alexandria at this time, Orestes, the Roman Governor at Alexandria, and Hypatia. In view of this triangular relationship, we shall recall briefly some of the important events just prior to and during the episcopate of Cyril and their relationship to the authority of the Roman Governor.

On October 12, 412, Theophilus, the Bishop at Alexandria, died, and six days later his nephew Cyril was elevated to the episcopate of Alexandria. From the outset the new bishop began to enforce with zeal the edicts of Theodosius I, the Roman Emperor, against the pagans, along with restrictions which he himself promulgated against the Jews and unorthodox Christians. He further began to encroach upon the jurisdiction which belonged to the civil authorities; that is, to the Roman Governor. It must be remembered that the population of the city of Alexandria in the fourth and fifth centuries of the Christian era consisted of a conglomeration of nationalities, creeds, and opinions, and that nowhere in the Empire did the Romans find a city so difficult to rule as Alexandria. The people were quick-witted and quick-tempered, and we read of numerous clashes, street fights, and tumults, not only between the citizenry and the soldiers, but also between the different classes of citizens themselves. There were frequent riots between the Jews and the Christians on the one hand and the pagans and the Christians on the other. The Christian population did little or nothing to quiet these people, but even added one more controversial topic for them to quarrel about. Consequently we find that the edicts and promulgations of Cyril not only caused strife among the people but aroused the anger of the Roman Governor, Orestes, the one person who stood in the way of the complete usurpation of the civil authority by Bishop Cyril. Friction continued between these two until there was a definite break in their relations.

Because of her intimacy with Orestes, many of the Christians charged that Hypatia was to blame, at least in part, for the lack of a reconciliation between Orestes and Cyril. Socrates states that some of them, whose ringleader was named Peter, a reader, driven on by a fierce and bigoted zeal, entered into a conspiracy against her. They followed her as she was returning home, dragged her from her carriage, and carried her to the church Cæsareum, where they stripped her and then murdered her with shells. They tore her body to pieces, took the mangled limbs to a place called Cinaron, and burned them with rice straws. This brutal murder happened, he says, under the tenth consulate of Honorius and the sixth of Theodosius in the month of March during Lent, so that the year of her death may be set as 415.

Socrates' report of Hypatia's death is corroborated not only by Suidas, but also by other historians such as Callistus,[1] the ecclesiastical historian, Philistorgus,[2] Hesychius[3] the Illustrious, and Malalus.[4] Damascius says that Cyril had vowed Hypatia's destruction, while Hesychius states that his envy was caused by her extraordinary wisdom and skill in astronomy. Damascius also relates that at one time Cyril, passing by the house of Hypatia, saw a great multitude, both men and women, some coming, some going, while others stayed. When he was told that this was Hypatia's house and that the purpose of the crowd of persons was to pay their respects to her, he vowed her destruction.

When we compare these statements, it would seem that Hypatia's death, or at least the occasion of it, was due to her friendship with Orestes. This friendship enraged the Christian populace because they felt that she prevented a reconciliation between Cyril and Orestes. We are also led to believe that the more sober-minded of the Christians yearned for a reconciliation between these two and that no doubt her death was ordered by Cyril.

Among the later writers on the subject there is a divergence of opinion. Toland lays the death of Hypatia directly at the feet of Cyril. Wolf[5], on the other hand, is inclined to believe that Cyril knew beforehand that the murder was being plotted but did nothing to prevent it. As to the causes of the murder, Wolf mentions her belief in paganism and her teaching of Neoplatonism, along with the practice of treating the mentally diseased with music, all of which might be considered as coming under the pale of the edicts of Theodosius I regarding pagan worship.

The present writer is inclined to follow Meyer part of the way in the interpretation of these events; that is, Hypatia was used as a sacrifice for a political or personal vengeance, possibly a political vengeance. Cyril and Orestes were at odds; both had made various reports to the Emperor, each one attempting to show that his actions were justified. On the other hand, Orestes was the one person who stood in the way of the complete assumption of the civil power by Cyril, and naturally Cyril was eager to use every incident which would embarrass Orestes. In the case of Hypatia's death it would seem that its underlying cause was not so much a struggle for the assumption of the

[1] *Nicephori Kallisti historia ecclesiastica Migne, Patrologiae Graecae*, Tome 147, Paris, 1856.

[2] *Ex ecclesiastici Philostorgii historia epitome confecta a Photio patriarcha*, H. Valesio interprete, Parisis, 1873.

[3] *Hesyychii Milesii Onomatologie que supersunt cum prolegomenis*, ed. J. Flacch, Lipsiae, 1882.

[4] Malalae, Johannus, *Chronographia ex recensione Ludovici Dindorfii*, Bonnae, 1831.

[5] Wolf, Stephan, *Hypatia, die Philosophin von Alexandrien*, Vienna, 1879.

civil authority, but rather a struggle of the Christian church against the pagan society of Alexandria. It must be remembered that although Orestes professed Christianity, the fact still remained that his profession was more one of policy than of faith. In all justice it would certainly seem that Cyril should be held at least indirectly responsible for her death. Certainly he could have prevented the mob's violence, if he had made the slightest effort.

Meyer feels the relation between Cyril and Synesius should be considered in investigating Hypatia's death. He is of the opinion that possibly there was an old difference between these two, and that her death was brought about by Cyril in order to settle this difference with Synesius. Meyer bases his conclusions on the contents of Epistle 12[1] of Synesius, in which he exhorts Cyril to go back to the Mother Church, from which he had been separated for a period of time for the expiation of sin. The present writer is of the opinion that Meyer has no justification for this assumption. Although we do not know the exact date of Synesius' death, it was probably between 412 and 414, and it must be remembered Cyril was not raised to the bishopric until late in the year 412. It is very probable that Epistle 12 was written before Cyril was made Bishop at Alexandria, though as a matter of fact we have no convincing evidence that the letter was written to Saint Cyril. Furthermore, there is no evidence to support the belief there ever existed any difference between Cyril and Synesius.

It has been stated above that little is known concerning Hypatia's early life. Consequently there is little on which to base our conclusions regarding her early education. It goes without saying that her father taught her in mathematics, astronomy, and science. Beyond this we do not know who her teachers were, but we may rest assured that, with an intellect as fertile as hers, she was not long satisfied with the narrow training in mathematics and astronomy. In order to understand the possible trend of her education it is necessary to take a look at the working of the Museum at Alexandria. The Museum had its origin in the efforts of Ptolemy Soter about 300 B. C., when he brought to the city of Alexandria all the philosophers and writers it was possible for him to obtain. To these he gave every encouragement possible, not only financial aid, but also in books and manuscripts from Greece. The later rulers of Egypt continued their support until the country came under Roman authority in 30 B. C. This ended the first period of intellectual activity, which is characterized as purely literary and scientific in nature. With the conquest of the country by the Romans, intellectual activity was again in the as-

[1] Synesii, *Opera quae extant omnia, Patrologiae, Graecae,* Tomus LXVI, Paris, 1864.

cendency and Roman, Greek, and Jewish scholars were again attracted to the city. This second school of thought was somewhat different from the first. We have an intermingling of nationalities with their varying philosophies and personalities all of which developed into the speculative philosophy of the Neoplatonist, the religious philosophy of the early Christian fathers, and the gnosticism of the Oriental philosophers. This second period of intellectual activity continued until about 642, when the city was destroyed by the Arabs. Considered as a whole, the Alexandrian School stood for learning and cosmopolitanism, for erudition rather than originality, and for a marked interest in all literary and scientific techniques. It was at the Museum that these philosophers, writers, and scientists gathered to lecture to their students and to converse with one another. Theon, Hypatia's father was director or fellow in the Museum, and it is reasonable to infer that Hypatia came into close contact with the leading educators and philosophers of Alexandria.

The question is frequently asked whether or not Hypatia studied at Athens. Here again we come to a point which has not been definitely decided. Suidas says she obtained part of her education there, or at least the passage has been so interpreted, for both Meyer and Hoche are of the opinion that Suidas has been misinterpreted on this point. Wolf states that Hypatia studied at Athens under Plutarch but Meyer again points out that this was highly improbable, as at the time Plutarch was lecturing at Athens, Hypatia was probably 30 years of age and was herself lecturing at Alexandria. Suidas also makes mention of the fact that she studied under another philosopher at Alexandria, but he does not identify this philosopher except to say that it was not Theon. Meyer thinks it might have been Plotinus. Regardless of how or where she received her education, we do know that she received a thorough training in arts, literature, science, and philosophy under the most competent teachers of the time.

It was with this training that she succeeded to the leadership of the Neoplatonic School at Alexandria. The exact date at which she assumed control of the school is not known, but Suidas informs us that she flourished under Arcadius, who was Emperor of the Eastern Roman Empire from 395 to 408. We are naturally led to ask two questions regarding her teaching: first, what was her ability as a teacher? second, what was the nature of her teaching? The first question is much simpler than the second, although there are sufficient facts relating to the nature of her teaching to enable us to draw a fairly definite conclusion.

All the contemporary and later writers on this period testify to the high reputation of her work as a teacher. Each one attributes an

extraordinary eloquence and an agreeable discourse to her lectures. Suidas speaks highly of her teaching methods, while Synesius in one letter praises her voice and in another mentions that her philosophy was carried to other lands. Socrates and Philistorgius tell us that not only the Egyptians, but students from other quarters of Europe, Asia, and Africa came to her classes until there was in reality a friendly traffic in intellectual subjects. Suidas states that, on account of her ability as a teacher and her personality, Orestes sought out her house to be trained in the art of public manners. Damascius states she far surpassed Isidorus as a philosopher, and it should be remembered that Damascius was a friend and pupil of Isidorus.

Among her disciples there are many well-known men other than Synesius. The names of these include Troillius, the teacher of the ecclesiastical historian Socrates, Euoptius, the brother of Synesius and probably the Bishop of Tolemais after the death of Synesius, Herculianus, Olympius, Hesychius, and finally Herocles the successor of Hypatia in the Platonic School at Alexandria.

From her teaching position she expounded the philosophy of the Neoplatonic School and her fame rests primarily upon the manner in which she conducted this school. In her teaching she no doubt lectured not only on philosophy as we know it today, but also included the scientific subjects of mathematics, astronomy, and the subject of physics as known at the time. She was apparently well versed in astronomy, since Suidas tells us that she excelled her father in this field. We may also assume that she taught the rudiments of mechanics, since there is a reference in one of Synesius' letters to an astrolabe which she constructed, and in another letter Synesius requests Hypatia to make a hydroscope for him.

Neoplatonism, as a philosophic system of thought, had its inception during the second century of the Christian era. It was built up from the remains of many of the systems of philosophies of ancient Greece and became a religion for many of the heathens, who could no longer believe in the old gods of Olympus. The Neoplatonist believed in a supreme being or power, which was the Absolute or One of the system. This supreme power was mystic, remote, and unapproachable in a direct fashion by finite beings. Hence there existed between man and the Absolute lesser gods or agencies. The first in this series was Nous or Thought, which was emanated by the Absolute as an image of itself. Below Nous there existed the triad of Souls, which pervaded all of the material universe, and all of those beings with which it is peopled are a direct emanation from the triad of Souls. Matter or material things were thought of as belonging to an evil category, while

the triad of Souls belonged to a pure category. Man, a mixture of the material and the spiritual, has the power by indulging in self-discipline and subjugation of the senses, to lift himself to a level where he may receive from the Absolute a revelation of divine realities. Once man has caught a glimpse of this vision, he is able to free himself entirely from the thralldom of matter.

It should be noted that the development was from a higher to a lower or descending series. Since each series participated in the one above it, there was also a turning back, where the soul by an ascending process was able to return to the Absolute. The object of life, when the soul was perfectly free, was to rise by the practice of virtue from the category of matter to the higher category of intelligible realities. There were purifying virtues, which disciplined the soul till it became capable of union with the Absolute.

We have no writings of Hypatia, but we may rest assured that she at least subscribed to the general principles of Neoplatonism. Plotinus' works show that he succeeded in contempt of bodily cares and needs, and we find the same thing to be true with Hypatia. No doubt Hypatia's use of logic, mathematics, and the exact sciences gave her a discipline which kept her and her pupils from going too far in the superstitions and speculations of some members of this group of thinkers. Synesius in his speech before the Arcadians, acknowledges the purely subjective character of the different attributes which are conceived of by man as belonging to the divine nature. He also felt a wholesome reticence in his attempts to reach towards the Incomprehensible. He believed in the Trinity of Plotinus, but did not assign to the World-soul the creating or animating of the entire universe. He thought occasional supernatural communications between God and the human soul were possible, and he also believed that man was able to purify his soul to such an extent that he would be able to elevate the imagination to a point where it would be possible for him to share in the ecstacy of the upper light. He believed that the final goal aimed at in life was a pure and tranquil state of mind, undistracted by fierce passions, gross appetites, or the demands of worldly affairs. It would be reasonable to assume that these tenets of Synesius' faith were inculcated in him by his beloved teacher Hypatia.

In considering the writings of Hypatia we have but little information to fall back on. Suidas is the only historian to give us any information concerning her writings. He gives us the names of three: a commentary on the *Arithmetica* of Diophantus of Alexandria, a commentary on the *Conics* of Apollonius of Pergassus, and a commentary on the *Astronomical Canon* of Ptolemy. None of these are extant at the present time.

We are naturally led to the question why Hypatia, a student of philosophy, a teacher of renown, and the leader of the Neoplatonic School at Alexandria, left only three works and those three purely mathematical or astronomical. The answer is probably that Suidas quoted the writings of Hypatia as given by Hesychius, who for some reason gives an account only of the Astro-Mathematical works of Hypatia. It is rather difficult for us to believe that with approximately twenty years of teaching she would produce not more than three works, and those three commentaries. So we are led to the conclusion that Hypatia did leave other writings, which were probably lost in the destruction of the library at Alexandria, and that these works were principally philosophic in nature. It is true that both Halma and Montucla[1] make mention of other works of Hypatia; Halma in particular says she left behind "beaucoup d'ecrits". At the present time it is impossible to determine from what source Halma obtained this information, and it is more than probable this is only a conjecture on his part.

With the passing of Hypatia we have no other woman mathematician of importance until late in the Middle Ages. Although we have no definite information to indicate that she exerted any great influence on the development of mathematics or science in general, nevertheless she certainly passed on to her scholars and followers a discipline and restraint which were carried over to a later period. It is possible that the effects of her teachings have been lost sight of, since any works she might have left behind were certainly lost when the Arabs destroyed the Library at Alexandria in 640.

[1] Montucla, J. F., *Histoire des Mathématiques*, Tome I, Part I, Liv. V, Paris, 1799.

Due to causes that need not be explained here it seemed advisable to the Editor and Manager of the MAGAZINE to omit (a) the department of *Mathematical World News* from the October issue and (b) the *Problem Department* from this, the November issue. The latter department will be included, as usual, in the December and future issues.

A History of American Mathematical Journals

By Benjamin F. Finkel
Drury College

(Continued from October, 1940, issue)

Let then

CASE SECOND, $b=2$.

The given equations become

$$\Sigma^m \cdot a^i = 2^{n'+1}c^{p'},$$
$$\Sigma^n \cdot b^i = 2^{n+1}-1 = a^{m''}c^{p''},$$
$$\Sigma^p \cdot c^i = 2^{n'''}a^{m'''},$$

m is necessarily odd.

$$2^{n'+1}c^{p'} = \frac{a^{m+1}-1}{a-1} = \frac{a^{(m+1)/2}-1}{a-1}(a^{(m+1)/2}+1).$$

1. Let $\dfrac{m+1}{2}$ be even.

Then $$\frac{a^{(m+1)/2}-1}{2^2} = \left(\frac{a^{(m+1)/4}+1}{2}\right)\left(\frac{a^{(m+1)/4}-1}{2}\right) = wh.$$

Therefore, $$\frac{a^{(m+1)/2}-1}{4}+\tfrac{1}{2} = \frac{a^{(m+1)/2}+1}{4} = wh. \text{ is impossible.}$$

So that $$\frac{a^{(m+1)/2}+1}{c} = wh.$$

and $$\frac{a^{(m+1)/2}+1}{c} - \frac{2}{c} = \frac{a^{(m+1)/2}-1}{c} = wh. \text{ is impossible.}$$

Therefore, $$a^{(m+1)/2}+1 = 2cp',$$

and $$\frac{a^{(m+1)/2}-1}{a-1} = 2^{n'} = (a^{(m+1)/4}+1)\left(\frac{a^{(m+1)/4}-1}{a-1}\right).$$

Hence, $$a^{(m+1)/4}+1=2^d,$$

and $$a^{(m+1)/4}-1=2^d-2=2^{n'-d}(a-1),$$

or $$n'-d=0,\quad \frac{m+1}{4}=1,\quad m=3.$$

$$2^{n'}=a+1,\quad a=2^{n'}-1.$$

1a. Let $n'''>0$. p must be odd.

And $$2^{n'''}a^{m'''}=\frac{c^{p+1}-1}{c-1}=\left(\frac{c^{(p+1)/2}-1}{c-1}\right)\cdot(c^{(p+1)/2}+1).$$

1a′. Let $\dfrac{p+1}{2}$ be even.

Then $$\frac{c^{(p+1)/2}-1}{2^2}=\left(\frac{c^{(p+1)/4}+1}{2}\right):\left(\frac{c^{(p+1)/4}-1}{2}\right)=wh.$$

Therefore, $$\frac{c^{(p+1)/4}-1}{c-1}=2^{n'''-1}=(c^{(p+1)/4}+1)\left(\frac{c^{(p+1)/4}-1}{c-1}\right)$$

and $$c^{(p+1)/2}+1=2a^{m'''}.$$

Hence, $$c^{(p+1)/4}+1=2^e,\quad c^{(p+1)/4}-1=2^e-2=2^{n'''-e-1}(c-1).$$

Therefore, $$e=n'''-1,\quad c=2^{n'''-1}-1,\quad p=3.$$

But we had $$c^{p'}=\frac{a^{(m+1)/2}+1}{2}=2^{2n'-1}-2^{n'}+1.$$

Therefore, $$(2^{n'''-1}-1)p'=2^{2n'-1}-2^{n'}+1.$$

If p' were odd, this would give $n'''-1=1$, or $n'=1$, both of which are impossible. Therefore, p' is even, and being less than p or 3.

$$p'=2.$$

Therefore, $$2^{2n'''-2}-2^{n'''-2}=2^{2n'-1}-2^{n'}.$$

Hence, $$n'=n'''-2.$$

Therefore, $2^{2n'''-2}=2^{2n'''-5}$, which is absurd.

Then $\dfrac{p+1}{4}$ is not even.

1a″. Let $\dfrac{p+1}{2}$ be odd and greater than 1.

In this case, $\dfrac{c^{(p+1)/2}-1}{c-1} \div 2 = wh.$ is impossible,

and $\dfrac{c^{(p+1)/2}+1}{c-1} = a^{m'''}, \quad c^{(p+1)/2}+1 = 2^{n'''}, \quad$ unless $\; p=1,$

but $$\frac{c^{(p+1)/2}+1}{c+1} = wh.$$

Therefore, $$c+1=2^{n_1}, \quad c=2^{n_1}-1.$$

Hence, $(2^{n_1}-1)^{(p+1)/2}+1=2^{n'''}$, impossible, unless $n_1=n'''$ and $p=1$.

1a'''. Let $\dfrac{p+1}{2}=1$, or $p=1$, and $p'=1$.

Then
$$c+1=2^{n''}a^{m'''},$$
$$c=2^{n'''}a^{m'''}-1,$$
$$2(2^{n'''}a^{m'''}-1)=a^{(m+1)/2}+1.$$

Here, if $m'''>0$, we have $\dfrac{3}{a}=wh.$ or $a=3$,

$$a=2^{n'}-1=3, \quad n'=2,$$
$$c=\frac{a^2+1}{2}=5,$$
$$c+1=6=2^{n'''}a^{m'''}, \quad n'''=1, \quad m'''=1, \quad m''=2,$$
$$n=n'+n'''=3, \quad 3^{n+1}-1=15=a^{m''}c^{p''}=3^2=9,$$

which is absurd.

But if $\quad m'''=0$, we have $c=2^{n'''}-1$,
$$a^{m''}=a^m=2^{n+1}-1=(2^{n'}-1)^m=(2^{n'}-1)^2.$$

Therefore, $n'=n+1=1$, which is impossible.

Therefore, n''' cannot be greater than zero.

1b. Let $\quad n'''=0$, p must be even.
$$n'=n, \; 2^{n+1}-1=a^{m''}c^{p''},$$
$$=(2^n-1)^{m''}c^{p''}.$$

Therefore, if $m''>0$, $\quad \dfrac{2^{n+1}-1}{2^n-1}=wh.$ which is impossible,

unless $$n=1, \quad \text{or} \quad a=1;$$

so that $$m''=0, \text{ and } m'''=3,$$

$$2^{n+1}-1=c^{p''}=2a+1.$$

But $$c^{p'}=\frac{a^2+1}{2}\ .$$

Therefore, $$8c^{p'}=4a^2+4=c^{2p''}-2cp''+5;$$

therefore, $$\frac{5}{c}=wh. \quad c=5,$$

$$8\cdot 5^{p'-1}=5^{2p''-1}-2\cdot 5^{p''-1}+1.$$

If $p''=1$, $8\cdot 5^{p'-1}=5-1=4$, which is impossible.

If $p''=1$, $7=5^{2p''-1}-2\cdot 5^{p''-1}$, which is impossible; therefore, $8\cdot 5^{p'-1}=5^{2p''-1}-2\cdot 5^{p''-1}+1$ cannot be satisfied,

and $$\frac{m+1}{2} \text{ cannot be even.}$$

2. Let $\dfrac{m+1}{2}$ be odd and greater than unity.

Then $$\frac{a^{(m+1)/2}-1}{a-1}=c^{p'}, \quad \text{where} \quad p'>0,$$

and $$a^{(m+1)/2}+1=2^{n'+1}.$$

But $$\frac{a^{(m+1)/2}+1}{a+1}=wh. \quad \text{Therefore, } a+1=2^d,$$

$$a=2^d-1, \quad \text{and} \quad (2^d-1)^{(m+1)/2}+1=2^{n'+1}.$$

Therefore, $d=n'+1$, and $\dfrac{m+1}{2}=1$, $m=1$, contrary to the present hypothesis.

3. Let $$\frac{m+1}{2}=1; \quad \text{or} \quad m=1.$$

$$a+1=2^{n'+1}c^{p'},$$

$$2^{n+1}-1=a^{m''}c^{p''},$$

$$\Sigma^p\cdot c^i=2^{n'''}a^{m'''}.$$

3a. Let $n'''>0$, p must be odd,

and $$2^{n'''}a^{m'''}=\frac{c^{p+1}-1}{c-1}=(c^{(p+1)/2}+1)\left(\frac{c^{(p+1)/2}-1}{c-1}\right).$$

3a′. Let $\dfrac{p+1}{2}$ be even.

Then $$\frac{c^{(p+1)/2}-1}{c-1}=2^{n'''-1},\quad\text{and}\quad c^{(p+1)/2}+1=2a^{n'''}=2a;$$

$$c^{(p+1)/4}+1=2^e,$$
$$c^{(p+1)/4}-1=2^e-2,$$
$$c^{(p+1)/2}-1=2^{2e}-2^{e+1}=2^{n'''-1}(c-1).$$

Therefore, $$e=n'''-1,\quad\text{and}\quad\frac{p+1}{4}=1,\quad p=3,$$

$$2a=c^2+1,\quad c+1=2^{n'''-1},$$
$$2a+2=2c^2+2^2=2^{n'+1}c^{p'}.\quad\text{Therefore, } p'=0,$$
$$2c^2+2^2=2^{n'+1},$$
$$c^2=2^{n'}-2,\quad\text{which is impossible.}$$

3a″. Let $\dfrac{p+1}{2}$ be odd and >1.

Then, $$\frac{c^{(p+1)/2}-1}{c-1}=a^{m'''}=a,\text{ and } c^{(p+1)/2}+1=2^{n'''}.$$

But $$\frac{c^{(p+1)/2}+1}{c+1}=wh.\quad\text{Therefore, } c+1=2^d;$$

$$(2^d-1)^{(p+1)/2}+1=2^{n'''},\quad n'''=d,\quad\frac{p+1}{2}=1,$$

contrary to hypothesis.

3*a*‴. Let $\dfrac{p+1}{2}=1$, or $p=1$.

$$c+1=2^{n'''}a^{m'''}.$$

3a$_1$‴. Let $m'''>0$ or $=1$, $m''=0$, $p''=1$, $p'=0$.

$$c+1=2^{n'''}a,$$

$$a+1=2^{n'+1},$$

$$c=2^{n+1}-1=2^{n'''}a-1,$$

$$a=\frac{2^{n+1}}{2^{n'''}},$$

which is impossible, since a is odd.

$3a_{11}'''$. Let $m'''=0$, $m''=1$.

$$a=2^{n'+1}c^{p'}-1=\frac{2^{n+1}-1}{c^{p''}},$$

$$2^{n'+1}c^{p}-c^{p''}=2^{n+1}-1=2^{n'+1}c-c^{p''}.$$

Therefore, $p''=1$.

And
$$c=\frac{2^{n+1}-1}{2^{n'+1}-1};$$

$$a=2^{n'+1}-1,$$

$$2^{n+1}-1=ac,$$

$$c=2^{n'''}-1,$$

$$2^{n+1}-1=2^{n+1}-2^{n'+1}-2^{n'''}+1,$$

$$1=2^{n'-2}+2^{n'''-1}.$$

Therefore, $n'''=1$, and $c=2-1=1$, which is impossible.

3b. Lastly, let $n'''=0$, p must be even.

$$m'''=1,\quad m''=0,\quad n'=n,$$

$$a+1=2^{n+1}c^{p'},$$

$$2^{n+1}-1=c^{p''},$$

$$\Sigma^{p}\cdot c^{i}=a=2^{n+1}c^{p'}-1. \quad \text{Therefore, } p'=0.$$

$$a+1=2^{n+1}=c^{p''}+1,$$

$$a=c^{p''}, \text{ which is impossible.}$$

Therefore, there can be no perfect number of the form $a^m b^n c^p$.

Following this article, pp. 277-282, are a few Diophantine problems with their solutions, by William Lenheart, of York, Pa.,—a gentleman who solved some quite difficult problems in that subject.

Pages 283-286 are occupied by the solution of a problem which was proposed in the *Scientific Journal**, of 1818. The solution is by Professor Strong of Rutgers College, New Brunswick, New Jersey.

On pages 287-289 is a biography of Lagrange, by Σ. We infer that Σ was Samuel Ward, 3rd. Pages 290-303 are given to a dialogue, *Horæ Decerptæ* written by Σ. On pages 304-305, Σ gives a very laudable review of an investigation of the motion of solids on surfaces, by Henry James Anderson, M. D., Professor of Mathematics in Columbia College, New York. Brief notices are given, pp. 305-307, of books and periodicals published at that time. Twenty-six new problems, pp. 308-311, are proposed for solution.

By the request of Mr. Samuel Ward, 3rd, New York, the editor proposes for consideration the following problem proposed by Mr. Ward and for a correct solution of which the proposer offers $100.00.

It is required to determine all the oscillations of a hollow hemi-ellipsoidal cup, and a given quantity of a homogeneous fluid contained in it, when the cup rocks without sliding about the shorter axis, upon a fixed horizontal plane.

This number concludes with a list of all contributors to and correspondents of, the *Mathematical Diary* from No. 1 to No. XIII inclusive.

With the publication of this number, the publication of the *Diary* ceased.

The dialogue referred to was written by Mr. Ward, and in it, he exhibits Dr. H. J. Anderson of Columbia College in a rediculous light.

David S. Hart, M. D., of Stonington, Conn., a gentleman well versed in Diophantine Analysis and well informed in many lines of thought has this to say in reference to the discontinuance of the *Diary*.† "Mr. Samuel Ward, 3rd, a recent graduate of Columbia College, had in part the management of the last number. In it he caused to be inserted a Dialogue, written by himself wherein he exhibits in a rediculous light Dr. Henry J. Anderson, then Mathematical Professor in Columbia College.... Both the Editor and Dr. Anderson were highly indignant at this performance. The parties met at the mathematical book-store of James Ryan, and high words passed between the parties and their friends. The result was the complete breaking up of the *Diary*, which was probably not intended nor anticipated by Mr. Ward."

Mr. Ward edited J. R. Young's *Elimentary Treatise on Algebra* which was published in Philadelphia in 1832. His father and grand-

*The publishers of this journal were Kirk and Mercein.

†See Vol. II. No. 5, (Sept. 1875) of the *Analyst*—Edited and published by Joel E Hendricks.

father bore the same name and hence he signed his name Samuel Ward, 3rd.

Vol. I, containing Nos. I-VIII, contained 316 pp.; Vol. II, containing Nos. IX-XIII, contains 316 pp.

The price of the *Mathematical Diary* was $1.00 per year in advance.

Bolton's Catalogue of Scientific Periodicals, 1665-1895 gives the following Libraries which possess copies of the *Mathematical Diary*:

Boston Public Library; Boston Arthanæum; Harvard University Library; Hamilton College Library, Clinton, N. Y.; Yale University Library; Cornell Library; Columbia Library; and the New York Public Library.

Vol. I and Nos. I and II of Vol. II are in the Library of the University of Pennsylvania, also a complete set. No. XIII is in the Library of the Am. Phil. Society, of Philadelphia. Dr. Artemas Martin, Washington, D. C., and the writer each own complete sets.

THE MATHEMATICAL COMPANION.

The next venture in the field of American mathematical journalism was *The Mathematical Companion*.

The following is its title page copied from the front cover of an eight page pamphlet in the Library of Harvard University arranged as nearly as possible in imitation of the original:

THE
MATHEMATICAL COMPANION;
containing
NEW RESEARCHES AND IMPROVEMENTS,
in the
MATHEMATICS;
with
COLLECTIONS OF QUESTIONS
Proposed and Resolved by
INGENIOUS CORRESPONDENTS

IN HALF YEARLY NUMBERS.

CONDUCTED BY JOHN D. WILLIAMS

——o——

New York:
Published by John D. Williams
No. 220 Bowery

——o——

R. Wauchope, Printer, No. 11 Spruce Street.

——o——

1828.
Price—Fifty Cents.

The inside of the front cover contains the following:

NOTICE TO CORRESPONDENTS.

—o—

1. All communications for the *Companion* must be post paid, and directed to the editor of the *Mathematical Companion*, 220 Bowery, New York.

2. New Questions must be accompanied with their solutions.

3. No. 1 will be published on the first day of May next, No. 2 on the first of September, and thereafter the numbers will be published regularly, on the first of January, and on the first of July.

TO SUBS TO SUBSCRIBERS CRIBERS

1. Subscribers who advance one dollar shall have the numbers of the *Companion* for one year, forwarded with the greatest punctuality and dispatch.

2. All orders for the *Mathematical Companion* must be post paid if sent by mail and direct to J. D. Williams, No. 220 Bowery.

Ready for the press, and will be shortly published by the Editor, A Key to Daboll's Arithmetic—Price, 50 cents. Also an *Elementary Treatise on Algebra*.

N. B. Whoever gives the best solution to the Prize Question, is requested to name some Mathematical Book valued at Five Dollars and it shall be sent to him as soon as it can be obtained.

J. D. WILLIAMS.

Page one contains the following:

MATH MATHEMATICAL QUESTIONS
SELECTED

By John D. Williams.

The list contains 36 questions, the 36th being a Prize Question, the list of questions occupying pages 1 to 5.

Pages 1-5 contain the following thirty-six questions:

MATHEMATICAL QUESTIONS
SELECTED

By John D. Williams.

1st. Suppose a vessel in the form of the frustum of a cone, the ratio of whose top and bottom diameters are as 5 to 3; the perpendicular depth 12 inches; what must the said diameters be for the said vessel

to contain 20 gallons; and supposing it hold 40 gallons what would be the length of the parts added to the greater and lesser ends: with a general rule? Also a geometrical rule is required.

This question is of a similar nature to the last question in Bonnycastle's *Mensuration*. (By J. D. Williams.)

2nd. Two masons A and B, jointly perform a piece of work in (a) days. Now if the sum of the days in which they could each perform the work separately be multiplied by the days in which A alone (A working quicker than B) could have done it, the product will be (b), in what time could each have performed the work alone? (By L. L.)

3rd. A and B have a certain number of dollars, A's dollars more the square root of B's make 86; and on the contrary, B's dollars more the square root A's make 34, required the number of dollars each had.

4th. A and B have a certain number of dollars, say A to B multiply the square root of your dollars by mine and the number will be $180, and say B to A multiply the square root of your dollars by mine and it will be 150 dollars. Required the number of dollars of each.

5th. Given the diameters of three opaque spherical bodies, equal 35, 40, and 45 feet, respectively, at 150, 160, and 170 feet distance from each other. Required, at what distance from each ball a luminous spherical body whose diameter is 10 feet, may be so placed as to enlighten the most surface of the other three when all their centers lie in the same horizontal plane. (By J. D. Williams.)

6th. Required the length of a woodbine which laps round a conical pole from the bottom to the top; distance between each lap 13 inches; length of the pole 45 feet; diameter at bottom 24 inches and at top 5 inches; required the distance that a boy must travel to unwind it keeping at its extremity. Note, the woodbine at rising makes an angle of 45 degrees with the base of the pole.

7th. A spherical body perfectly elastic, to fall freely, by the force of its own gravity, from the height of 1,000 (a) yards, on the summit of a firm rock, whose inclination to the plane of the horizon is 15 degrees. The greatest velocity, the distance of its second descent on the same plane, and the whole time of its continuance in motion are required with the general method of investigation. (By J. A.)

8th. If a sphere of copper etc. of one foot diameter, were to be beaten into a circular plate $^1/_{20}$ of an inch thick, require the diameter of the same. (By J. H.)

9th. Given three spherical balls of 1, 2, and 4 oz. avoirdupois weight to be suspended by three silk chords of proper length and strength, so that they will vibrate seconds when a well adjusted barom-

eter stands at $29^1/_2$ inches and thermometer at 55 degrees temperature; and supposing this state of air to remain, and the three pendulums being lifted up to make an angle of 5 degrees from their perpendicular direction with the horizon, and then let swinging. Query, the number of vibrations each will make, and the time in motion; with a general formula. (By W.)

10th. There is a punch bowl in the form of a frustum of a spheroid; the depth is equal to $^2/_9$ of the whole length of the spheroid, and the diameter at the top of the bowl 30 inches, also the conjugate diameter of the spheroid is $^1/_2$ the transverse. It is required to tell how many rounds 10 men may take with a conical glass whose diameter at the top is $3^1/_2$ inches and depth is 2.

11th. A coppersmith sold a concave globe of copper (at 1s. 9d. per cubic inch) the outside diameter of which was 33 inches, and its immersion was $^1/_3$ when in common water. Required the value of the globe. (By S. B.)

12th. The minute hand of a watch performs the 60th part of a revolution in the time that pendulum of 67 inches in length makes 44 vibrations: how much will it vary from the truth in 24 hours? (By J. B.)

13th. A bullet being let fall eternally in this manner; the first minute 20 miles, the second 19 miles, the third minute $18^1/_{20}$ miles, and so onward forever in the same geometrical progression; how far will it fall in a whole eternity?

14th. Required (the least) two right angled triangles in whole numbers, with a common hypothenuse, so that the square of the greater leg of the one may exceed the square of the greater leg of the other by a square whole number; with general method of investigation. (By John D. Williams.)

15th. Given the length, the diameter at the greater end, and the diameter at the lesser of the frustum of a cone $=30$, 20, and 10 inches, respectively; which is to be cut lengthwise in such a manner that the breadth of the section may be everywhere equal to four inches. Required the nature of the section and the solidity of either part: with a general method of investigation. (By R. I.)

16th. If a ball that is perfectly elastic be let fall perpendicularly upon an exceeding hard horizontal plane; it is required from what height it ought to fall, so that the time of its discent to the said plane, and its ascent after reflection to a given point in the perpendicular line shall be a minimum. (By T. A.)

17th. Suppose a person lets fall a heavy ball from the top of a tower. It was observed to reach the bottom of the tower the instant

of time a cannon ball was discharged from a cannon at the distance of 200 yards from the tower. Also, the time from the beginning of the body's falling to the time the person at the top of the tower heard the report of the cannon was 4 seconds. Required the tower's height. (By W.)

18th. Into a vessel filled with rain water, suppose there be put a conical foot of dry wood, with the lesser end downwards, and its axis perpendicular to the surface of the water; and when a plane passing through the center of gravity, parallel to the base, coincides with the water's surface, it is found to rest in equilibrium. Required the quantity that will run over and the specific gravity of the cone. (By T. B.)

19th. Given, ax^2 plus bx equal to c, to find the value of x without dividing by a or completing the square.

20th. Question by T. R. and reproduced by Swillmai, A. M. A gentleman has standing on the side wall of his park, 2 towers, as A 76 feet high, and B 57; their distance asunder being 114 feet, and between the towers at 44, 56 feet from the tower A is a gate, from which a walk leads up; perpendicular from the wall, an unknown distance in the said perpendicular to a fountain, and from thence is the said perpendicular continued, 100 feet to an obelisk; then is the distances from the fountain to the top of each tower and the obelisk all equal to each other. Query, the distance from the fountain to each tower, and the height of the obelisk. This problem is impossible; but a problem is said to be solved when it is demonstrated to be impossible. Also point out the defects, etc.

22nd. Given AB and BC in the triangle ABC and the segment of the base DE made by two lines BD and BE, drawn from the vertical angle to the base—the angle ABE, equal to CBD equal to 90 degrees. Required to construct the triangle. (By J. D. Williams.)

23rd. Suppose a small circular plate of uniform density and thickness touches a sphere at a point indefinitely near its center. It is required to determine the time of the oscillations of the plate, supposing that its particles are attracted by the sphere with forces which vary inversely as the square of their distances from the center, the plate being indefinetly thin.

24th. A given, perfectly elastic, sphere is let fall upon 3 given perfectly elastic spheres, from a given height, 50 yards. It is required to determine its motion after the stroke. Required a general theorem. (By John D. Williams.)

25th. The dimensions of a given cone, are viz.: height 300 feet, base diameter 60 feet, top diameter 1 foot; now a rope one-half inch in

diameter is to be wound round it in a spherical manner until the surface is completely covered to the top. Required the length of the rope, and the distance a boy must travel to unwind it keeping at its extremity. All the solutions of this question, heretofore, which I have seen, are incorrect. (By John D. Williams.)

26th. A cord of given length, and uniform density and thickness, is suspended from one of its extremities, and then raised up to a given height and let go. It is required to determine the nature of the curve described by its lower end in its descent admitting it to be inflexible. (By J. D. Williams.)

Question. By John M'Canley, Liverpool, England.

27th. The hour and minute hands of a clock are in the same straight line, and in a position perpendicular to the horizon at 6 o'clock. Query, are they ever in a straight line exactly parallel to the horizon when in a right line? and, if not, what is the least angle which they can make with the horizontal line, when thus situated? (From Nash; Diary.)

Question. By N. G. Miles, New York.

28th. A person would have a fish-pond in his garden, in the form of an ellipse, the area to contain 301.5936 perches, and the ratio of the parameter to the sum of its axes as 4 to 15. Required the dimensions of the ellipse. (From same.)

Question. By E. of New York.

29th. Find two numbers such that the sum of their squares may be the sum of a square and cube, and the sum of their cubes, the difference of a square and cube. (From same.)

Question. By Mr. J. Phillips of Harlem.

30th. If a paraboloid, whose diameter is 8, and altitude $10^1/_2$, be rolled on a smooth horizontal plane, in such a manner that the axis produced may always pass through the same point, it is required to determine the area of the curve generated by the point of contact. (From same.)

Question. By John Capp, Harrisburg, Pa.

31st. A gentleman purchasing a rectangular garden, whose perimeter is to be 100 yards, agreed to pay one dollar for every yard of length of the ground, and four dollars for every yard of its breadth. It is required the minimum price of a square yard of this ground? (From same.)

Question. By W. Marrat, Liverpool, England.

32nd. Let a cone the altitude of which is 6, and base diameter 2 feet, be filled with water, the sides of the cone being indefinitely thin,

and of the same specific gravity as water; let this cone be suspended by its vertex, and made to vibrate like a pendulum; if an orifice of $^1/_{10}$ of an inch in diameter be opened in its base, in what time will the vessel vibrate? after the water has been flowing for one minute? (From same.)

By Mr. J. H. Swale, of Liverpool, England.

33rd. The right lines EM, EF, EK, and the point A are given in position, and C is a given point in EM: Draw AC, AM, intersecting EK, EF, in D, Q, and join D, Q. Then, if any line whatever, as AKE be drawn through A, meeting EK, EF, at K, F, the point of concourse P, of CF, MK, will always be in the line DQ. The demonstration of this local theorem is required.

Question. By Mr. S. Jones, of Liverpool, N. Y.

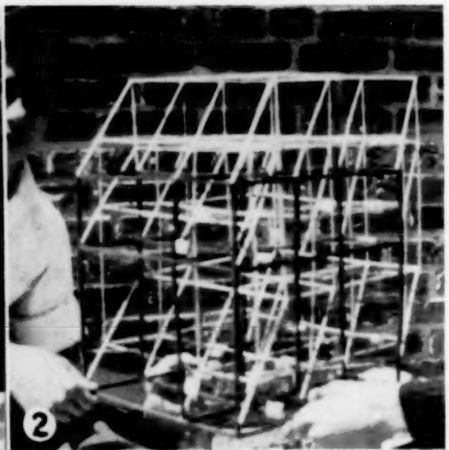
34th. If a given circle revolves perpendicularly upon a horizontal plane and if from a given point in the track of the circle a tangent be drawn, and produced to meet the vertical diameter in Q, required the equation and quadrature of the curve which is the locus of Q?

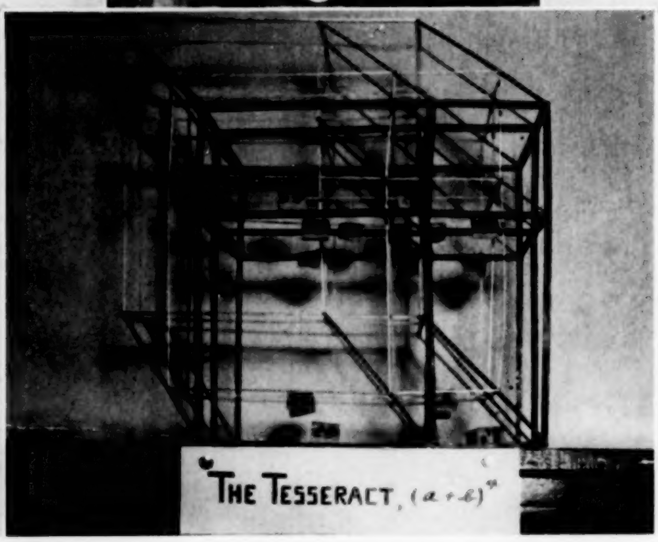
Question. By J. D. Williams, N. Y.

35th. Required to find a rational right angled triangle, such that the sum of the squares of the straight lines drawn from the acute angles to the middle of the opposite sides will be a cube, the area equal to the cube of one of the legs, the sum of the squares of the straight lines bisecting the acute angles and terminating in the opposite legs, a double square, and the perimeter increased by the sums of the diameter of the inscribed circle, and that of the inscribed square, shall be a square.

Prize Question. By J. D. Williams, N. Y.

36th. Suppose a perfectly elastic uniform circular hoop, of very small thickness and density is suspended by a given pivot on its plane vertical. It is required to determine its form supposing all its parts to be acted upon by a uniform gravity.





1. *Roscoe and Jack Stick and Glue Artists*

2. *The box that grew a Fourth Dimension*

3. *The Goofy Geometers*

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

The Tesseract, $(a+b)^4$

A DEMONSTRATION OF THE BINOMIAL THEOREM IN FOURTH DIMENSIONAL GEOMETRY

By HARRIET B. HERBERT
Edmunds High School, Sumter, S. C.

Two boys in one of my plane geometry classes in Sumter, S. C., recently constructed a model of $(a+b)^4$. The following is their own version of how it came about.

"During the past year, our geometry class under Miss Herbert, began selecting from certain topics suggested by Miss Herbert, work to be done at home or in class. Among these topics were listed several such as the study of measurement of ancient times, making a mathematical quiz program for radio, making a balsa model of $(a+b)^3$, study of magic squares, etc. Roscoe Newton chose from this group the making of $(a+b)^3$. He built this model and brought it to school. The base was a square, $(a+b)$ on each side. The line a was about four inches long, b was nine inches. Each face of the cube was divided into $a^2+2ab+b^2$, that is, a small square, two rectangles, and a large square; while the whole cube was $a^3+3a^2b+3ab^2+b^3$, or, a small cube, three tall blocks, three flat ones, and a large cube. Roscoe painted the short a ends of the balsa sticks brown and the long b ends green.

"Roscoe liked this work with airplane sticks and glue so much better than the idea of writing a paper on Thales that he asked for another job like it. So Miss Herbert said, "Why not carry the thing to the fourth dimension?" She had to explain that a fourth dimension figure in our space of three dimensions is a representation rather than an actual figure. Almost anybody can draw a box, that is, represent three dimensions on two-dimensional paper. Just as the third dimension of the box is shown in perspective, so we must represent the fourth

dimension of our model. Being strangers in the fourth dimension, we had to experiment a bit before attempting anything fancy like $(a+b)^4$. Roscoe and I, Jack Raffield, with Miss Herbert acting as guide, began constructing a simple tesseract to try out this plan of showing fourth dimension by perspective. Making a cube first, then at each vertex attaching a stick put on obliquely to represent the fourth dimension, we held all these slanted sticks parallel to each other and joined their ends by pairs with horizontal and vertical sticks. We made the fourth dimension lines a little shorter than the others so as to carry out the idea of their being seen in perspective.

"Now, we were ready to start work on $(a+b)^4$. With the tesseract just finished as a pattern, we began work on adding the fourth dimension to the large $(a+b)^3$ that Roscoe had built and painted. Using lighter balsa than that of the original cube, we began adding a shortened $(a+b)$ as fourth dimension. These lines had to be all parallel, one from each vertex, and joined by pairs with vertical and horizontal lines which were exactly $(a+b)$ in length. When the whole tesseract was completed, we started putting in the inside lines of the fourth dimensional part of the figure. Placing our large tesseract so that the small cube, a^3, was at the upper right-hand corner of the original cube, we decided to complete a^4 there by taking a for the part of the oblique line which was joined on to each vertex of a^3. This meant that all of the oblique lines of the figure, then, must be divided into, first a, then b, (both a and b being foreshortened as above stated). The joining of these points that divided a from b completed the figure.

"Now, all we had left to do was to locate all parts of $(a+b)^4$. It was interesting to attempt to figure out where the different parts came in. There was no doubt as to a^4. We had planned its place first in the upper right-hand corner. There it was, easily recognizable as a small tesseract. Four lines, all a's, met at each vertex, three at right angles and the fourth oblique to the others. We wondered whether an a^2b of the cube had become a^3b or a^2b^2. This was easily determined by observing whether its dimension added obliquely was a short line, a, or a long line, b. The large cube, b^3, fooled us by turning out to be ab^3, when we looked there for b^4. The large tesseract, b^4, was nearby, though. We labeled them all, a^4, four a^3b's, six a^2b^2's, four ab^3's, and b^4. Then we took a picture of our model, $(a+b)^4$."

The whole class was interested, as were my solid geometry and trigonometry classes, in another demonstration of the binomial theorem which was best seen in the simple tesseract constructed as a pattern. We noticed, in considering its number of vertices, edges, etc., that for line, square, cube, and tesseract, $(2+1)^n$, where n is the dimension of

the figure, gives the number of these respective features. For the line, (2+1), had 2 end points and 1 line segment; the square, $(2+1)^2$, had 4 vertices, 4 edges, and 1 face; the cube, $(2+1)^3$, had 8 vertices, 12 edges, 6 faces, and 1 volume; and the tesseract, $(2+1)^4$, had 16 vertices, 32 edges, 24 faces, 8 volumes (six of these eight cubes being distorted because seen in perspective), and 1 hyper-volume (the entire tesseract). This, we find, is a previously known theorem.

The boys were spurred on throughout their making of $(a+b)^4$ and identifying of its parts by my telling them that I had never heard of anyone's attempting this particular thing and was not sure how it would come out.

The following is a list of American universities and colleges whose mathematics faculty members have submitted the largest number of papers for presentation at the meetings of the American Mathematical Society in the period from July, 1937, to September, 1940. The list is made up of those institutions whose mathematics faculties have each published more than one abstract. These statistics were obtained by examination of the lists of abstracts published in the Bulletin of the American Mathematical Society in the period named.*

Name of University or College	*Number of Published Abstracts*
Alabama, University of	14
Arkansas, University of	4
Adelphi College	4
Antioch College	3
Arizona, University of	9
Brooklyn College	24
Brown University	20
British Columbia, University of	5
Bryn Mawr College	2
Cornell	55
Chicago, University of	46
California Inst. of Tech.	45
Columbia University	44
California, University of (Berkeley)	31
California, Uni. of (Los Angeles)	29
Carnegie Inst. Tech.	10
Cooper Union	6
Colby College	6
Case School of Applied Science	13
Cincinnati, University of	12
Citadel, The	4
Duke	32
Dartmouth	2
Fenn College	3
Georgia Tech.	18
Harvard	69
Hunter College	10
Hofstra College	5
Haverford College	5
Illinois, University of	70
Illinois Inst. of Tech.	60
Institute for Advanced Study	22
Indiana University	9
Iowa State College	5
Iowa State, University of	4
Johns Hopkins	14
Kansas, University of	9
Kentucky, University of	12
Kenyon College	2
Kansas State College	2
Lehigh University	18
Louisiana State University	5
Louisiana Polytechnic Institute	4
Michigan, University of	72
Mass. Inst. of Tech.	30
Minn., University of	30

*These compilations were worked out by Mrs. Carrie Ables, one of the secretaries of the Louisiana State University Mathematics department.—(*Concluded on page 101*).

Mathematical World News

Edited by
L. J. Adams

In the June 14, 1940, number of *Science*, Professor G. A. Miller, University of Illinois, discusses the history of the terms ***associative law, commutative law***, and ***distributive law***. He calls attention to the lack of consistency in the usage of these items of nomenclature in the "Encyclopedia Brittanica" (1938) and also in Zassenhaus's "Lehrbuch der Gruppentheorie," volume 1 (1937).

Professor R. F. Grœsser, head of the department of mathematics at the university of Arizona, announces the appointment of Dr. Franz E. Hohn as instructor in mathematics at the University of Arizona. Dr. Hohn completed his doctorate this summer at the University of Illinois, and he will commence his duties in this department next month.

Professor Julian L. Coolidge was awarded the honorary degree of LL.D. by Harvard University at Commencement, 1940.

The Seventeenth Yearbook of The National Council of Teachers of Mathematics will be entitled "Compendium of Mathematical Applications". It is contemplated that the principal topics of Arithmetic, Algebra, Geometry and Trigonometry will be listed alphabetically, together with practical applications in some of the fields and vocations where the particular topic is useful. Since the book is being prepared as an aid for the teacher of elementary mathematics, problems will be of a type and at a level to appeal to the high school student. A cross-index will enable the teacher to survey the breadth of application in any particular field. The following committee has been appointed to prepare the Yearbook: L. W. Boyer, State Teachers College, Millersville, Pa.; Ruth O. Lane, University High School, Iowa City, Iowa; Nathan Lazar, Bronx High School of Science, New York, N. Y.; F. L. Wren, George Peabody College for Teachers, Nashville, Tenn.; Edwin G. Olds, Carnegie Institute of Technology, Pittsburgh, Pa., Chairman. However, no small committee could hope to have the breadth of education necessary to survey all fields and the above committee is no exception. Therefore, it earnestly solicits the aid of all friends of mathematics in discovering the most significant applications of elementary mathematics. Contributions and suggestions should be submitted to a member of the committee as soon as possible.

The mathematics section of the Southern California Junior College Conference was scheduled to meet at the University of Southern California in Los Angeles on Saturday, October 12, The program included:

1. *Some Applications of Mathematics in the Aircraft Industry.* Dr. Clauser, Douglas Aircraft Co., Inc.
2. *Challenge Problems.* Mr. Charles W. Trigg, Los Angeles City College.

(*Concluded from page* 99)

Name of University or College	*Number of Published Abstracts*	*Name of University or College*	*Number of Published Abstracts*
Missouri, University of	15	Queen's College	22
Maryland, University of	9	Queen's University (Canada)	2
McGill University	6	Rochester, University of	6
Mt. Allison University	5	Rutgers	6
Michigan State	2	Rice Institute	4
Mississippi State College	2	Reed College	2
Montana State	2	Stanford University	18
North Carolina, University of	11	Saskatchewan, University of	7
Northwestern University	44	Smith College	12
Notre Dame	20	Swarthmore College	2
New York University	16	St. Mary's College	2
New York, College of the City of	5	St. Francis, College of	2
North Carolina State College	7	St. Thomas, College of	2
Nebraska, University of	6	Texas, University of	24
New Mexico State College	3	Toronto, University of	14
New Hampshire, University of	3	Tulane University	7
Nevada, University of	2	Texas Tech	5
Ohio State	23	Virginia, University of	33
Oklahoma A. & M.	8	Vanderbilt	2
Oklahoma, University of	6	Wisconsin, University of	28
Oregon, University of	3	Washington State, University of	17
Oregon State College	2	Wells College	7
Ohio University	2	Wm. Penn College	4
Oberlin College	2	Washington University (Mo.)	4
Princeton	34	Woodrow Wilson Junior College	4
Pennsylvania, University of	11	Washington, State College of	3
Pennsylvania State	10	Wesleyan University	2
Purdue University	12	Yale	36
Pomona College	4	Yeshiva College	13
Phila. College Phar. and Science	2		

Bibliography and Reviews

Edited by
H. A. Simmons

The Application of Moving Axes Methods to the Geometry of Curves and Surfaces. By G. S. Mahajani. Aryabhushan Press, Poona, India, 1937. viii+60 pages.

This book is the result of an essay, written by the author as an undergraduate, which has been rounded out with numerous problems and examples to form a sort of monograph.

The basic problem may be stated as follows: Suppose a set of three mutually perpendicular axes were to move in some definite fashion, and for each position a point Q is to have a definite position with respect to these axes, then the differential properties of the locus of Q are to be found. In particular, the vertex of the three axes may move along a curve C, with the tangent, principal normal, and binormal of C at each point of P of C as the three axes, and Q may be related to P in each position in some definite fashion, the center of the osculating circle for example.

The author develops the fundamental formulas in the first chapter; he uses vectors, and while he goes somewhat into detail, his development requires considerable previous knowledge of vectors.

In the second chapter numerous applications to curves are treated with surprising ease, but a previous knowledge of the differential geometry of curves is desirable. Such topics as the Frenet formulas, evolutes, locus of the center of the osculating sphere, and Bertrand curves are discussed in a satisfactory manner.

The last two chapters are devoted to the relations between two sets of axes, each associated with each point of a curve on a given surface, and to the applications of these relations to curves on a surface. Many of the fundamental theorems of curves on a surface, such curves as asymptotic lines, lines of curvature, and geodesics, are obtained very easily from the basic relations.

A general treatment of these moving axes in most differential geometry texts is either inadequate or entirely missing, as is probably proper for an introductory course. However this treatment can well be used for additional work, or as a basis for advanced individual work.

Northwestern University. FRANK EDWIN WOOD.

College Algebra. By Charles H. Sisam. Henry Holt and Company, New York, 1940. xii+395 pages. $1.90.

The first eight chapters (124 pages) are devoted to the topics of elementary algebra, through *quadratic equations*. In view of this rather extensive treatment of these topics, the book is well suited for use by those students who have had only one year of *high school algebra*. The remainder of the text, containing fourteen chapters (221 pages), includes all of the material which has become traditional in American college algebras. Hence, the book is equally well suited for more advanced students of algebra.

This book is very well written, the introduction of each new idea being clearly illustrated by one or more simple examples. As a consequence of this commendable

practice, the text should prove to be readable even to a college freshman. The large number of good exercises in each section should make a strong appeal to those people who feel that at least some class room drill is indispensable in beginning courses. Also worthy of note is the treatment of *theory of equations* which is somewhat more extensive than usual, comprising two chapters (46 pages), and a final chapter on *infinite series* (23 pages), a topic which frequently is not included in college algebra texts.

Answers to the odd numbered problems are given in the back of the book. An appendix contains short tables for use with certain topics, such as *logarithms, interest and annuities*, etc.

University of Wisconsin at Milwaukee. ROSS H. BARDELL.

College Algebra. By P. R. Rider. New York, The Macmillan Company, 1940. ix+372 pages. $2.00.

The general scope and content of this textbook conform to the pattern prescribed by the usual freshman course in algebra. The early chapters constitute a review of elementary and intermediate topics. Then follow chapters on mathematical induction and the binomial theorem, progressions, complex numbers, theory of equations, logarithms, compound interest and annuities, permutations and combinations, probability, determinants, partial fractions, infinite series, and finite differences. In addition to a table of powers and roots, and a four-place table of logarithms, there are adequate tables for use in connection with the chapter on compound interest and annuities. Also included is an index and a list of answers to the odd-numbered exercises.

The book is carefully written and each chapter contains a liberal supply of graded problems. The chapter on theory of equations consists of rather more material than is to be found in many of the books of similar scope. Teachers interested in statistics will welcome the appearance in an algebra book of a chapter on finite differences, brief though it is.

Large and clear type, dignified pages, and a robust binding, contribute to an excellent format.

Northwestern University. JOHN F. KENNEY.

Mathematical Clubs and Recreations. By Samuel I. Jones. S. I. Jones Co., Nashville, Tenn., 1940. xiv+236 pages. $2.75.

This book is a deserving and welcome addition to the group of books on mathematical recreations such as previous works by the same author, works by Ball, Licks, and Philips. However, this book is much more than a work on mathematical recreations. It deals also, as the title states, with mathematical clubs. It is this part of the book which will be found most helpful to the mathematics teachers and students who are interested in founding or maintaining clubs either in secondary schools or colleges.

Part I (sixty-five pages) is headed *Mathematical Clubs*. This section discusses the history, purposes, and achievements of mathematics clubs in the United States. Sample constitutions are given and representative programs. Here the author considers the need of developing a real interest in the class room study of mathematics and he offers the club as one means of creating this interest. Along these lines he has some well chosen quotations from other articles, selected principally from *The Mathematics Teacher*.

Part II (one hundred thirty-five pages) is entitled *Mathematical Recreations* and is divided into fifteen subdivisions, several of which seem to overlap. Among the

most interesting of these are the articles on *Number Rhymes, Games, Puzzles* and *Riddles,* and *Bible Tests.* Here is a wealth of material for clubs, class room, and even for the *Sunday School* class.

Part III (twenty-six pages) is entitled *Solutions to Recreations.* This seems as a whole to be carefully compiled and satisfying.

The reviewer questions the historical accuracy of the statement on page 22, quoted in a sample club program, that the oldest book on mathematics in existence is the *Rhind Papyrus.* The repeated use of the abbreviation *Math* (no period) does not seem to give sufficient dignity to the subject. This is particularly noticeable when it appears in the same line with the word *Mathematics* written out in full as it does on page 23 under the heading *The Math Teacher's Aim.*

The reviewer doubts if the word *column* could be applied to the answer to *Brain Teaser,* No. 11, as given on page 218. There is a slight error in the answer to No. 12, page 218, as the reciprocal of 7 is .142857142857.142857142857···—and not 142857.

The general appearance of the volume is attractive. The square diagram on the front cover is one of the interesting details of the book. An index adds to its usability as a reference work.

University of North Dakota. R. C. STALEY.

A First Course in College Mathematics. By W. E. Anderson. Harper & Brothers, New York, 1939. $4.00.

This text covers the material in college algebra, trignometry, and analytic geometry that is usually taught in the Freshman year in college. Enough of the calculus is introduced to solve for maxima and minima of simple functions. About thirty pages in the latter part of the book are devoted to solid analytic geometry.

An effort is made to treat the subject matter as a whole. The idea of the function is stressed and is used as a coordinating link between the different types of subject matter. Deductive processes are utilized rather extensively. New concepts are consistently introduced in general form. It is doubtful if such procedure is desirable at this stage of the student's mathematical development.

The task of correlating the subject matter of algebra, trigonometry, and analytic geometry has been reasonably well carried out but the text possesses some of that choppy quality that is so universally found in works covering the three fields.

Exercises are plentiful and well chosen. The text is well written and printed, and figures are plentiful and apt.

Iowa State College. P. G. ROBINSON.

Three Copernican Treatises. Trans. by Edward Rosen. Number XXX of the Records of Civilization, Sources and Studies, Austin P. Evans, Editor. New York: Columbia University Press, 1939. xi+211 pages. $3.00.

No greater tribute can be paid to the astronomical theories of Eudoxus and Ptolemy than the simple statement of the fact that for fifteen hundred years after Ptolemy, observational astronomy did not attain the precision requiring a different theoretical explanation. Copernicus and Kepler, in his first astronomical work, desired a new theory chiefly to give a more evident harmony to the celestial universe.

The propriety of associating Copernicus with Kepler, Tycho Brahe, Galilei, and Newton in the evolution of astronomical theories is disputed since in some measure

Copernicus' developments, retaining explanation of the observed movements of the heavenly bodies by epicycles, retarded progress towards more correct considerations.

The harmonious universe expressed in the laws of Kepler and of Newton bears eloquent witness to the correctness of the intuitive feelings of Copernicus and Kepler that simpler mathematical formulation of the motions of the heavenly bodies were a necessity.

The contribution of Copernicus as a radical departure from the old theories and embodying many new points of view can only be appreciated by a study of his writings as related to the developments of his period.

To Mr. Edward Rosen and the Columbia University group of historians and scientists who are intelligently interested in the *Records of Civilization*, we are indebted for English accounts of the first astronomical contribution of Copernicus (1524) and more particularly for the *Commentariolus*. which was prepared by Copernicus about 1530 to explain his new theories. The *Commentariolus* does not give the final Copernican system, as published in 1543. However, this document, circulated in manuscript, introduced the essentials of the new theories to the astronomers of that day.

In addition to these two documents the *Narratio prima* of 1540, by Rheticus, expounding the new theories to the astronomer John Schöner in the form of a letter, is given in English translation. Undoubtedly this summary by an able disciple was scrutinized by Copernicus, and it presented a simplification satisfactory to him.

The documents given clearly indicate more modern points of view, explicitly stated, which constitute a vital part of the Copernican theory. The dominating position of the sun is affirmed by Copernicus (p. 139) in the statement, "The sun's rule in the realm of nature must be revived." The insistence upon the earth's motions and the effect of these motions on the recorded observations are also fundamental points of view, of importance in subsequent developments.

Copernicus' insistence upon the circular motions and upon uniform motion for each heavenly body undoubtedly simplified the mathematical application of the new theories but complicated further developments.

In this period a new mathematics was developing with great rapidity. Algebra made such progress that Vieta was able to introduce a consistent literal symbolism. For astronomers trigonometry and trigonometric tables made tremendous progress, made possible by Hindu-Arabic developments. These methods and materials combined with the new observations furnished by Tycho Brahe rendered the Ptolemaic theory and the modifications of Copernicus and of Tycho Brahe equally untenable. As Kepler put it much later, there were discrepancies of "8 seconds," sufficient to justify a wholly new astronomy. After Tycho the movements of the planets could no longer be harmonized with the old theories.

The Copernican theories sufficed to enable Erasmus Reinhold to produce by 1551 the Prutenic tables, which were much better than the Alphonsine tables.

The new theory also sufficed to enable Copernicus to predict that Venus and Mercury would be found, when improved means of observation were invented, to have phases like the moon. This was verified in 1609 by Galilei in one of the earliest contributions with the telescope.

One cannot claim that Copernicus was as great an observer as Tycho Brahe nor that he had the mathematical ability of Kepler, who calculated the paths of the planets from the numerical results of refined observations. These two men of superlative genius, Tycho Brahe and Kepler, met and reconstructed the universe, using the mathematical-astronomical developments of centuries. In this achievement, Copernicus had an honorable part.

This volume is particularly to be commended for the scholarly use of that wide range of material, notably the Arabic and medieval, which is necessary to the proper interpretation of Latin mathematical and astronomical treatises of the period of Copernicus. The precision attained in the translation of the technical terminology greatly increases the usefulness of this presentation of a difficult theory.

The translator has the further idea to attempt the publication in 1943 of the first complete English translation of the *De Revolutionibus*. With this done, a much wider public than at present might become capable of judging the contributions of Copernicus.

University of Michigan. LOUIS C. KARPINSKI.

Elementary College Mathematics. By Mackie and Hoyle, Ginn and Company, New York, 1940. ix+331 pages+78 pages of tables. $2.80.

This book represents another welcome departure from established methods of presenting elementary college mathematics. Its expressed aim is to combine the advantages of the so-called *unified* courses with those of the treatment by separate subjects: algebra, trigonometry, analytic geometry, calculus, etc. The method used for accomplishing this desirable synthesis is to start with a very thorough review of elementary algebra and to follow that up with a study of freshman college algebra. The student is then ready for linear functions and straight lines, and functions of second and higher degrees. Then follows a chapter on rates. In this way various topics, such as the quadratic equation, to mention but one of many, are considered from several different standpoints. Thereby there is attained one of the chief advantages of the unified treatment, a many-sided perspective, a grasp of a topic without loss due to excessive duplication, and yet with sufficient training in fundamentals and processes. The unifying process in this synthesis is accomplished largely by the student himself in referring back and forth through the book. This will be inspiring to the student and will serve to promote his mathematical growth.

There are several interesting minor teaching devices used in smoothing rough spots on the pathway to the *Calculus*. For example, in a very early chapter, there is a treatment of distance represented as an area under a curve,—well illustrated by graphs and diagrams; and there is also a well illustrated treatment of areas found approximately by means of trapezoids. Another example of such a device, used very early in the book, is in connection with the slope of a straight line, the slope being denoted by

$$\frac{\Delta y}{\Delta x}.$$

Another interesting and valuable device, used to circumvent long and technical, difficult, definitions, is the presenting of classifying tabulations. This rivets the attention at once on a host of common properties as well as distinguishing features. The student begins to see some of the inspiring mountains and valleys, instead of becoming bogged down in the swamps of formal definitions.

The chapter on Calculus goes into differentiation and integration of polynomials and simple types of algebraic functions by way of a study of rates, average rates and instantaneous rates, and of limits and elementary limiting processes. Naturally enough, integration is treated as the inverse of differentiation.

The calculus chapter is followed by a chapter on trigonometry which starts at once with the general angle and the general definitions of the trigonometric functions, and is followed by logarithms and the solution of triangle problems. A very good set of tables is provided at the end of the book.

A good innovation in regard to the exercises is the dividing of each set into three parts: part A being oral, part B being provided with answers, part C being without answers. There are many diagrams and graphs throughout the book, and the workmanship and printing is most excellent.

But one adverse comment, of a minor kind, might be made against the so-called alternative method of evaluating a third-order determinant, illustrated on page 118, by forming partial products along the principal diagonal and the secondary diagonal. Such an alternative method should not be offered without an accompanying warning that it is not valid in general.

The book represents a one-year course, for three hours per week. There is no treatment of central conic sections.

University of Florida. B. F. DOSTAL.

LITERATURE RECEIVED BY THE EDITORIAL BOARD DURING PERIOD JUNE 1, 1940 – NOVEMBER 1, 1940

1. *Bemerkungen über die Galoissche Gruppe einer Gleichung* (aus einem Briefe an Herrn B. L. van der Wærden). Von Michael Bauer in Budapest. Reprint from Mathematische Annalen, 114, 352, 1937.
2. *Considérations sur le problème de la durée du jeu.* Par Daniel Arany, Budapest, Hongrie. Reprint from The Tôhoku Mathematical Journal, Vol. 30, Nos. 1, 2, September, 1928.
3. *Der rechte Winkel und die Kubischen Konstruktionen.* Von R. Oblath in Budapest. Zeitschrift Für Mathematischen und Naturwissenschaftlichen Unterricht Aller Schulgattungen. 68 Jahrgang 1937. 7, Heft.
4. *Degree of Approximation by Polynomials in z and $1/z$.* By W. E. Sewell. Reprint from Duke Mathematical Journal, Vol. 4, No. 2, June, 1938.
5. *Degree of Approximation by Polynomials to Continuous Functions.* By W. E. Sewell. Reprint from Bulletin of the American Mathematical Society, February, 1935.
6. *Degree of Approximation by Polynomials—Problem α.* By W. E. Sewell. Reprint from the Proceedings of the National Academy of Sciences, Vol. 23, No. 9. pp. 491-493. Sept. 1937.
7. *Diophantine Equations of Degrees n.* By A. A. Aucoin. Reprint from the Bulletin of the American Mathematical Society. Vol. 46, No. 4, pp. 334-339, April, 1940.
8. *Existence Theorems for Solutions of Differential Equations of Non-Integral Order.* By Everett Pitcher and W. E. Sewell. Reprint from Bulletin of the American Mathematical Society. February, 1938.
9. *Generalized Derivatives and Approximation.* By W. E. Sewell. Reprint from the Proceedings of the National Academy of Sciences. Vol. 21, No. 5, pp. 255-258. May, 1935.
10. *General Formulas for the Number of Magic Squares Belonging to Certain Classes.* By Erich Stern, Amsterdam (The Netherlands). (Translated from the German by W. R. Transue, Lehigh University). Reprint from the American Mathematical Monthly. Vol. XLVI, No. 9. November, 1939.

11. *Jackson Summation of the Faber Development.* By W. E. Sewell. Reprint from Bulletin of the American Mathematical Society. February, 1939.

12. *Le problème des parcours.* Par D. Arany, Budapest, Hongrie. Reprint from The Tôhoku Mathematical Journal, Vol. 37. June, 1933.

13. *Note on Degree of Trigonometric and Polynomial Approximation to an Analytic Function.* By J. L. Walsh and W. E. Sewell. Reprint from Bulletin of the American Mathematical Society. December, 1938.

14. *Note on the Relation Between Continuity and Degree of Polynomial Approximation in the Complex Domain.* By J. L. Walsh and W. E. Sewell. Reprint from Bulletin of the American Mathematical Society. August, 1937.

15. *A Note on the Relation Between Integral and Tchebycheff Approximation by Polynomials in the Complex Domain.* By W. E. Sewell. Reprint from Bulletin of the American Mathematical Society. June, 1937.

16. *Note sur "Le troisième problème du jeu".* Par M. Daniel Arany à Budapest. Acta, Litterarum Ac Scientiarum, Regiae Universitatis Hungaricae Francisco-Iosephinae. Tom II, Fasc. I. 1924.

17. *On the Modulus of the Derivative of a Polynomial.* By W. E. Sewell. Reprint from Bulletin of the American Mathematical Society. October, 1936.

18. *On the Polynomial Derivative Constant for an Ellipse.* By W. E. Sewell. Reprint from American Mathematical Monthly. Vol. XLIV, No. 9. November, 1937.

19 *Paarerzeugung Beim Beta-Zerfall.* Von L. Tisza. Physikalische Zeitschrift der Sowjetunion. Band 11, Heft 4, 1937.

20. *Project for the Computation of Mathematical Tables.* Federal Works Agency, N. Y. Sponsored by the Bureau of Standards, Washington, D. C.

21. *Sur la Généralisation du Problème de la Durée du jeu pour Trois Joueurs.* Par D. Arany,Budapest. Estratto dagli Atti del Congresso Internazionale dei Matematici Bologna, 3-10 settembre, 1928—VI.

22. *The Derivative of a Polynomial on Further Arcs of the Complex Domain.* By W. E. Sewell. Reprint from the American Mathematical Monthly. Vol. XLVI, No. 10. December, 1939.

23. *Über die Zusammensetzung algebraischer Zahlkörper.* Von Michael Bauer in Budapest. Acta, Litterarum Ac Scientiarum Regiae Universitatis Hungaricae Francisco-Iosephinae. Tom. IX. Fasc. IV. 9. II. 1940.

24. *Über diophantische Gleichungen der Form* $n!=x^p\pm y^p$ *und* $n!\pm m!=x^p$. Von Paul Erdös in Manchester und Richard Oblath in Budapest. Acta, Litterarum Ac Scientiarum Regiae Universitatis Hungaricae Francisco-Iosephinae. Tom. VIII. Fasc. IV. 20. VII. 1937.

25. *Zur Theorie der Kreiskörper.* Von Michael Bauer in Budapest. Acta, Litterarum Ac Scientiarum Regiae Universitatis Hungaricae Francisco-Iosephinae. Tom. IX. Fasc. II. 10. V. 1939.